

1. If Bob takes 6 hours to build 4 houses, he will take  $6 \times 3$  hours to build  $4 \times 3 = 12$  houses. The answer is  $\boxed{18}$ .

2. Here is a somewhat elegant way to do the calculation:

$$\begin{aligned} \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} &= \left(\frac{2}{2} - \frac{1}{2}\right) + \left(\frac{3}{6} - \frac{2}{6}\right) + \left(\frac{4}{12} - \frac{3}{12}\right) + \left(\frac{5}{20} - \frac{4}{20}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) \\ &= \frac{1}{1} - \frac{1}{5} \\ &= \boxed{\frac{4}{5}}. \end{aligned}$$

3. For  $x$ -intercept, set  $y = 0$ . Then  $2x = 170$ , which gives  $x = 85$ . For  $y$ -intercept, set  $x = 0$ . Then  $5y = 170$ , i.e.  $y = 34$ . The answer is  $85 + 34 = \boxed{119}$ .

4. From 1st April to 19th November, there are 30 (April) + 31(May) + 30 (June) + 31 (July) + 31 (August) + 30 (September) + 31 (October) + 18 (November) days. The total number of days is 232. Dividing by 7,  $232 = 33 \times 7 + 1$ . So there are 33 complete weeks and one leftover day. If the day on 19th November is Saturday, then the day on April 1st is Saturday - 1 =  $\boxed{\text{Friday}}$ .

5. If Mr. Popper wants to take the maximum number of penguins possible, the other 9 people must take the minimum possible number of penguins. Each person takes at least one penguin, and no two take the same number. So we let the first person take 1 penguin, the second person take 2, ... and the 9th person take 9 penguins. The total number of penguins taken by these 9 people is  $1 + 2 + \dots + 9 = 45$ . The remaining number of penguins is  $78 - 45 = \boxed{33}$ . This is the maximum number of penguins Mr. Popper can take.

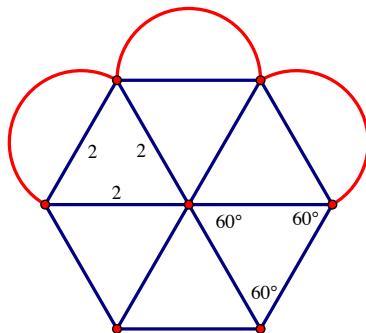
6. The letters  $M, A, T, H, I$  have a vertical axis of symmetry.  $M, A$  and  $T$  occur twice, while  $H$  and  $I$  occur once. So there are 8 such letters out of 11. Answer is  $\boxed{\frac{8}{11}}$ .

7. Since Eve wants to read Alice and Bob's notes, she must sit in between them. So let us permute the other 4 and three empty seats. Denoting everyone by first letter, we can write  $CDFGSSS$ , where  $S$  are the empty seats. The number of ways to permute these are  $\frac{7!}{3!}$ . Finally, given the three seats  $S...S...S$ , we can have either  $A...E...B$  or  $B...E...A$  so that Eve is between the two of them. Thus there are only two possible ways. So the answer is  $\frac{7!}{3!} \times 2 = \boxed{1680}$ .

8. We will count the number of occurrences of 8 in the units digit and tens digit separately. 8 appears in the units digit 47 times, namely  $\overline{00}8, \overline{01}8, \overline{02}8, \overline{03}8, \dots, \overline{47}8$  (read the numbers formed by the overlined digits). 8 appears in the tens digit 41 times, namely  $\overline{08}0, \overline{08}1, \overline{08}2, \dots, \overline{48}0$  (read the numbers formed by the overlined digits). So the answer is  $48+41 = \boxed{89}$ .

9. The following is a picture of the figure. We have not drawn all the semicircles.

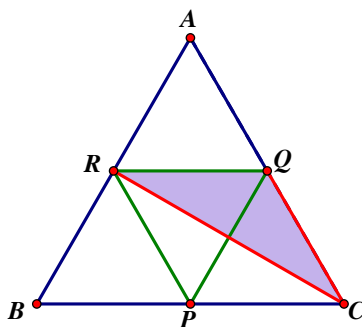
When we draw the segments from the center of the regular hexagon to the vertices, the  $360^\circ$  is divided into 6 equal parts, each being  $60^\circ$ .



Also, the segments joining the center to the vertices are equal in length, thus forming isosceles triangles. So the triangle formed by the center and two adjacent vertices is equilateral of side length 2. Its area is  $2^2 \frac{\sqrt{3}}{4} = \sqrt{3}$ . And the semicircle on the outer side of the hexagon has radius 1, hence it has area  $\frac{\pi}{2}(1)^2 = \frac{\pi}{2}$ . There are a total of 6 semicircles and 6 triangles. Thus the answer is  $6 \times \frac{\pi}{2} + 6 \times \sqrt{3} = 3\pi + 6\sqrt{3}$ .

10.  $291 = a^2 - b^2 = (a + b)(a - b)$ . The prime factorization of 291 is  $291 = 3 \times 97$ . Since  $a, b > 0$  we have  $a + b > a - b$ .  $a, b$  are non-consecutive, thus  $a - b > 1$ . Since  $291 = (a + b)(a - b)$  is a factorization of 291, let us look at possible factorizations of 291.  $291 = 291 \times 1 = 97 \times 3$ . But  $a - b > 1$ , so we cannot have  $a + b = 291$  and  $a - b = 1$ . The only possible way is to have  $a + b = 97$  and  $a - b = 3$ . Solving for  $a$  and  $b$  (add the two equations together), we get  $a = 50, b = 47$ .

11. The following is a picture of the figure. We have not drawn all the triangles.



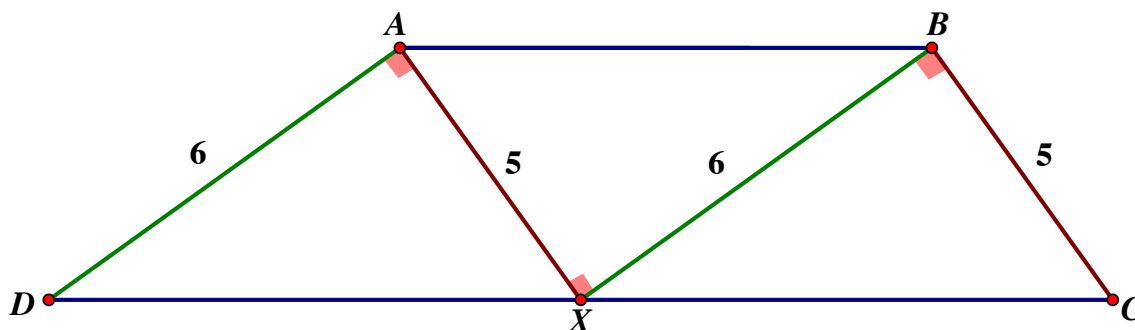
A is at the same distance from  $QR$  as  $P$ . Since the heights are same,  $Area[PQR] = Area[AQR]$ . Moreover, because  $P, B$  and  $C$  lie on a line parallel to  $QR$ , they all are at the same height from  $QR$ . Thus  $Area[BQR] = Area[PQR] = Area[CQR]$ .

Taking the side  $QR$ , we have obtained 3 triangles of equal area, namely  $AQR, BQR, CQR$ . Similarly, taking side  $PQ$ , we will obtain other triangles, namely  $APQ, BPQ, CPQ$ . And for  $PR$ , we get  $APR, BPR, CPR$ . Counting these 9 triangles and the original triangle  $PQR$ , the answer is  $\boxed{10}$ .

There are no other triangles because any other triangle will have at least two vertices from the larger triangle,  $ABC$ . No matter what the last vertex is taken to be, the triangle will have area at least twice that of  $PQR$ .

Bonus: The same triangles have equal area even if the original triangle is not equilateral.

12. The first 15 binary emulating number will look like the first 15 binary numbers, 0001, 0010, 0011, ... 1111. (Even though 0000 is not a binary emulating number, we add at the beginning for the sake of completeness.) There there are nice patters for each digit. For the units digit the pattern is 0,1,0,1,... For the second digit the patter is 0,0,1,1,0,0,1,1,... For the third digit, the pattern is 0,0,0,0,1,1,1,1,0,0,0,0,1,1,1,1. And for the fourth digit the pattern is 0,0,0,0,1,1,1,1. When we add up all the respective digits, we get the number 8888 in base three. Of course, this is not a true base three number since you have to carry over during addition, but since you want to convert it into decimal it does not matter if you keep it in this form. Converting back to decimal,  $8 \times 8^3 + 8 \times 3^2 + 8 \times 3 + 8 = 8(27+9+3+1) = 8 \times 40 = \boxed{320}$ .
13. If we choose a set of these numbers, any two of these numbers cannot have a common prime divisor. So each of these numbers must have a unique prime divisor. If we formed a new set where, instead of these numbers, we chose the unique prime divisor in them, this new set would have the same property that no two numbers have a common divisor. This new set consists only of primes. So the answer is the set of primes less than 30. The number of such primes is 10. But our set can also consist of 1, because 1 does not have a divisor greater than 1. So the maximum number is  $\boxed{11}$ .
14. The following is a picture of the figure. Segments of the same color are parallel.



Because of the series of right angles, we have  $AD$  parallel to  $BX$  and  $AX$  parallel to  $BC$ . Since  $AB$  and  $CD$  are the bases of the trapezoid, they are parallel too. Hence  $ABXD$  and

$ABCX$  are parallelograms (opposite sides parallel). It follows that  $\overline{AX} = \overline{BC} = 5$  and  $\overline{BX} = \overline{AD} = 6$ .

So  $\text{Area}[ABCD] = \text{Area}[AXD] + \text{Area}[ABX] + \text{Area}[BXC]$ . Each of these three triangles is right angled with base lengths 5 and 6. Hence  $\text{Area}[ABCD] = 3 \times \left(\frac{5 \times 6}{2}\right) = \boxed{45}$ .

15. If Alice wins the game, she must be the winner of the last round. Also, Bob can win at most 3 rounds.

We make cases based on the number of rounds won by Bob. If he wins 3 rounds, then the total number of rounds is  $4 + 3 = 7$ . Bob wins 3 rounds out of the first 6. The number of ways we can choose three rounds out of the first 6 is  $\binom{6}{3} = 20$ . Alice wins the other 3 rounds, and also the 7th round.

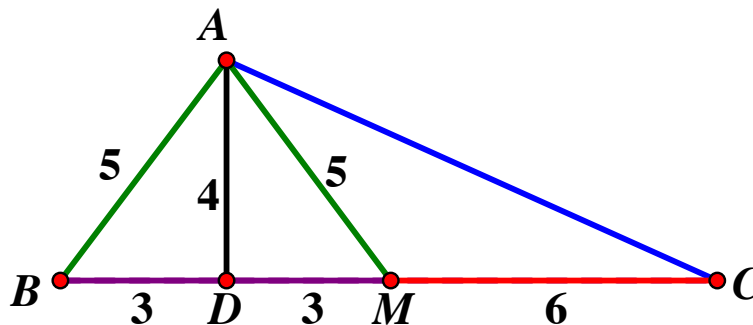
If Bob wins two rounds, the total number of rounds is  $4 + 2 = 6$ . Bob wins 2 rounds out of the first 5. The number of ways we can choose two rounds out of the first 5 is  $\binom{5}{2} = 10$ . Alice wins the other 3 rounds, and also the 6th round.

If Bob wins one round, the total number of rounds is  $4 + 1 = 5$ . Bob wins 1 rounds out of the first 4. The number of ways we can choose one round out of the first 4 is  $\binom{4}{1} = 4$ .

If Bob wins no round, then there is exactly 1 way in which Alice wins all four rounds.

The total is  $20 + 10 + 4 + 1 = \boxed{35}$  ways.

16. The following is a picture of the figure.



Since  $\overline{AB} = \overline{AM} = 5$ ,  $\triangle ABM$  is an isosceles triangle. So if  $D$  is the midpoint of  $BM$ , then  $AD \perp BM$ .

Since  $M$  is the midpoint of  $BC$ ,  $\overline{BM} = \overline{MC} = \frac{\overline{BC}}{2} = 6$ . And since  $D$  is the midpoint of  $BM$ ,  $\overline{BD} = \overline{DM} = 3$ .

By Pythagoras theorem,  $\overline{AD}^2 = \overline{AM}^2 - \overline{DM}^2 = 5^2 - 3^2 = 4^2$ . Thus  $\overline{AD} = 4$ . With base as  $BC$  and height as  $AD$ ,  $\text{Area}[ABC] = \frac{4 \times 12}{2} = 24$ .

17. Since 42, 43, 44, 45, 46 are good we can write each of them as  $5x + 8y$  for some positive  $x$  and  $y$ . If  $42 = 5x + 8y$ , then  $42 + 5k = 5(x + k) + 8y$ , which means that all numbers of the form

$42 + 5k$  are good for positive integer  $k$ . Hence 42, 47, 52, 57, ... are all good. Similarly, adding multiples of 5 to the other numbers, we have 43, 48, 53, ... good, 44, 49, 54, ... good and so on. So all numbers greater than or equal to 42 are good.

So we must check all numbers less than 42. With a little trial and error  $41 = 25 + 16 = 5 \times 5 + 8 \times 2$ . So 41 is good.

40 is not good. Every positive multiple of 8 that is less than 40 is 8, 16, 24, 32. You cannot add a positive multiple of 5 to any of these to get 40. Since every number greater than 40 is good, the largest number that is not good must be  $\boxed{40}$ .

18. We need some observations:

If a square has even side length, then it contains an even number of squares. Because the odd and even numbers are arranged in an alternating pattern, a square with even side length will contain an even number of odd numbers. Hence the sum of the numbers inside such a square will be even.

If a square has odd side length, it is not always true that it is an odd square. For instance, if you take the  $3 \times 3$  square with top-left corner as 2, then the sum of the numbers is  $2 + 3 + 4 + 9 + 10 + 11 + 16 + 17 + 18 = 90$ , which is even. In particular, if you take a square with odd side length and which has an even number in its top left corner, then it is not an odd square (such squares always have an even number of odd numbers inside it. if you look at such squares in the figure you will see why.) On the other hand, if you take a square with odd side length and which has an odd number in its top left corner, then it is an odd square. Thus, our task is to count all squares with odd length which have an odd number in its top left corner.

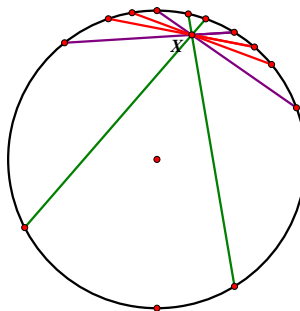
This is basic counting. There is one  $7 \times 7$  square. There are five  $5 \times 5$  squares, with top left corners 1, 3, 9, 15 and 17. There are thirteen  $3 \times 3$  squares, with top left corners 1, 3, 5, 9, 11, 15, 17, 19, 23, 25, 29, 31, 33. And there are obviously twenty-five  $1 \times 1$  squares (just count the odd numbers).

The answer is  $1 + 5 + 13 + 25 = \boxed{44}$ .

(Surprisingly, there is a pattern to these numbers.  $1 = 0^2 + 1^2$ .  $5 = 1^2 + 2^2$ .  $13 = 2^2 + 3^2$ .  $25 = 3^2 + 4^2$ . It can help you predict the answer in squares larger than  $7 \times 7$ .)

19. By the law of constant multiplication, any chord  $MN$  passing through  $X$  satisfies  $\overline{MX} \cdot \overline{NXPX} \cdot \overline{QX} = 6 \times 2 = 12$ . The possible factorizations of 12 are  $12 = 12 \times 1 = 6 \times 2 = 4 \times 3 = 3 \times 4 = 2 \times 6 = 1 \times 12$ . There are 6 possible factorizations. If we want 6 euphonic chords, then each chord corresponds exactly with one of the above factorizations.

In particular, there must be two chords of length  $12 + 1 = 13$ . So the diameter of the circle must exceed 13. Since the radius is integer, the diameter must be even. The smallest possible diameter is 14. Hence the radius must be  $\boxed{7}$ . A little thought shows that for such a circle we can draw the 6 euphonic chords. (Or maybe the picture below will convince you: the green chords are of length  $12 + 1 = 13$ . The purple chords are of length  $6 + 2 = 8$ . And the red chords are of length  $3 + 4 = 7$ ).



20. Note  $324000 = 2^5 \cdot 3^4 \cdot 5^3$ . This game is identical to one in which we have three stacks of sticks of size 3, 4, and 5 and may take away a positive number of sticks from only one stack each round. This game is commonly known as Nim (find more information online), and to determine from which pile we should take sticks, we want to look at the binary expressions and obtain a Nim-sum of zero. This yields piles of  $11_2$ ,  $100_2$ , and  $101_2$ , and to make the Nim-sum zero, we require  $11_2 \rightarrow 01_2$ . This corresponds to taking two sticks from the pile of size 3, or dividing by  $5^2 = \boxed{25}$ .