1. What is the largest number of five dollar footlongs Jimmy can buy with 88 dollars?

   **Answer: 17**

   **Solution:** If Jimmy buys 17 five dollar foot longs, he will have $88 - 17 \cdot 5 = 3$ dollars left over. Thus, the maximum number of footlongs is $\boxed{17}$.

2. Austin, Derwin, and Sylvia are deciding on roles for BMT 2021. There must be a single Tournament Director and a single Head Problem Writer, but one person cannot take on both roles. In how many ways can the roles be assigned to Austin, Derwin, and Sylvia?

   **Answer: 6**

   **Solution:** There are 3 choices for who should be Tournament Director. Once a Tournament Director has been chosen, there are 2 remaining choices for who should be Head Problem Writer, since the Tournament Director and Head Problem Writer cannot be the same person. Thus, there are $3 \cdot 2 = \boxed{6}$ ways to assign the roles.

3. Sofia has 7 unique shirts. How many ways can she place 2 shirts into a suitcase, where the order in which Sofia places the shirts into the suitcase does not matter?

   **Answer: 21**

   **Solution:** There are 7 options for choosing the first shirt and 6 options for choosing the second shirt. The shirts can be reordered in 2 ways, so there are $\frac{7 \cdot 6}{2} = \boxed{21}$ combinations.

4. Compute the sum of the prime factors of 2021.

   **Answer: 90**

   **Solution:** Factoring 2021, we have that $2021 = 43 \cdot 47$, so the sum is $43 + 47 = \boxed{90}$.

5. A sphere has volume $36\pi$ cubic feet. If its radius increases by 100%, then its volume increases by $a\pi$ cubic feet. Compute $a$.

   **Answer: 252**

   **Solution:** If the radius of a sphere increases by 100%, this means it has doubled in length. As a result, the volume of the sphere increases by a factor of 8 and thus by $8 \cdot 36\pi - 36\pi = 252\pi$ cubic feet, so $a = \boxed{252}$.

6. The full price of a movie ticket is \$10, but a matinee ticket to the same movie costs only 70% of the full price. If 30% of the tickets sold for the movie are matinee tickets, and the total revenue from movie tickets is \$1001, compute the total number of tickets sold.

   **Answer: 110**

   **Solution:** Let $x$ be the number of full priced movie tickets and $y$ be the number of matinee tickets. Then we can set up the equations $10x + (10 \cdot 0.7)y = 1001$ and $y = \frac{3}{10}(x+y) = \frac{3}{10}x + \frac{3}{10}y$, or $7y = 3x$. Solving, we get $x = 77$ and $y = 33$, so $x + y = \boxed{110}$.

7. Anisa rolls a fair six-sided die twice. The probability that the value Anisa rolls the second time is greater than or equal to the value Anisa rolls the first time can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

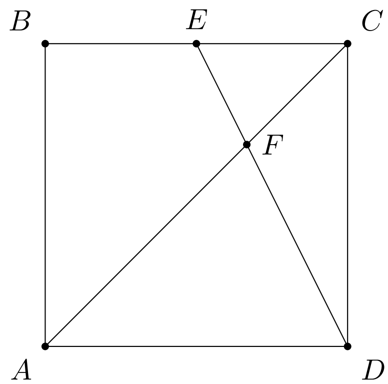
   **Answer: 19**

   **Solution:** With probability $\frac{1}{6}$, the two rolls are equal. Therefore, with probability $\frac{5}{6}$, the two rolls are not equal, so with probability $\frac{5}{12}$ the second roll is larger. Therefore, the probability the second roll is the greater than or equal to the first roll is $\frac{5}{12} + \frac{1}{6} = \frac{7}{12}$, so $m + n = \boxed{19}$.

8. Square $ABCD$ has side length $AB = 6$. Let point $E$ be the midpoint of $\overline{BC}$. Line segments $\overline{AC}$ and $\overline{DE}$ intersect at point $F$. Compute the area of quadrilateral $ABEF$.

**Answer: 15**

**Solution:**




Note that the area of quadrilateral $ABEF$ is the area of $\triangle ABC$ subtracted by the area of $\triangle CEF$. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 6 \cdot 6 = 18$, so we need to compute the area of $\triangle CEF$.

Also note that $\triangle AFD$ and $\triangle CFE$ are similar, so $\frac{DF}{EF} = \frac{AD}{CE} = 2$. Thus, since $\triangle CEF$ and $\triangle CFD$ share the same height, the ratio of their areas is the ratio of the bases, so the ratio of the area of $\triangle CEF$ to the area of $\triangle CFD$ is the ratio of $EF$ to $DF$, which is $\frac{1}{2}$. Thus, the area of $\triangle CEF$ is $\frac{1}{3}$ the area of $\triangle CDE$. The area of $\triangle CDE$ is $\frac{6 \cdot 3}{2} = 9$, so the area of $\triangle CEF$ is $\frac{1}{3} \cdot 9 = 3$. Thus, the area of $ABEF$ is $18 - 3 = \boxed{15}$.

9. Justine has a large bag of candy. She splits the candy equally between herself and her 4 friends, but she needs to discard three candies before dividing so that everyone gets an equal number of candies. Justine then splits her share of the candy between herself and her two siblings, but she needs to discard one candy before dividing so that she and her siblings get an equal number of candies. If Justine had instead split all of the candy that was originally in the large bag between herself and 14 of her classmates, what is the fewest number of candies that she would need to discard before dividing so that Justine and her 14 classmates get an equal number of candies?

**Answer: 8**

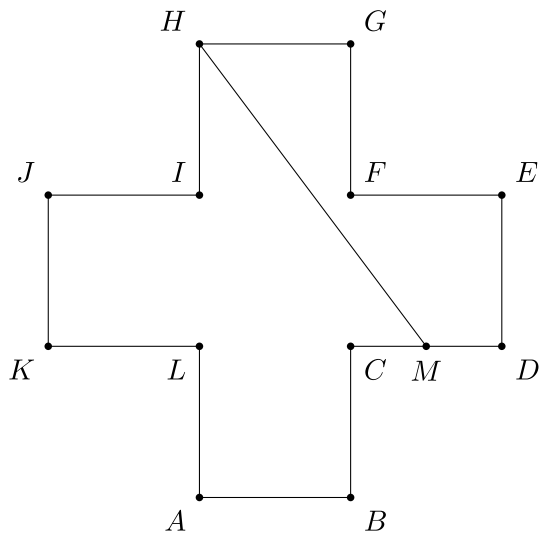
**Solution:** Let $k$ be the number of candies that one of Justine's siblings get. Then the number of candies that Justine recieved after splitting the candy among her and her 4 friends is $3k+1$, and then the number of candies that were initially in the large bag would be $5(3k+1)+3 = 15k+8$. Thus, in order to split the candies equally among 15 people, Justine needs to discard $\boxed{8}$ candies.

10. For some positive integers $a$ and $b$, $a^2 - b^2 = 400$. If $a$ is even, compute $a$.

**Answer: 52**

**Solution:** Note that $a^2 - b^2 = (a+b)(a-b)$ is the product of two even integers. We can observe this by noticing that $a + b$ and $a - b$ differ by $2b$, and thus are either both even or both odd; since 400 is even, we know they cannot both be odd. We can then write $400 = 200 \cdot 2 = 100 \cdot 4 = 50 \cdot 8 = 40 \cdot 10 = 20 \cdot 20$, and find the corresponding values of $a$ and $b$. Of these, only $100 \cdot 4$ yields an even value for $a$, which is $\frac{100+4}{2} = \boxed{52}$.

11. Let $ABCDEFGHIJKL$ be the equilateral dodecagon shown below, and each angle is either $90°$ or $270°$. Let $M$ be the midpoint of $\overline{CD}$, and suppose $\overline{HM}$ splits the dodecagon into two regions. The ratio of the area of the larger region to the area of the smaller region can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m+n$.




**Answer: 10**

**Solution:** Observe that the desired area is the union of square $IJKL$, square $LABC$, and right triangle $\triangle HML$. Note that $ML = \frac{CD}{2} + LC$ and that $LH = LI + IH$. Letting the side length of the dodecagon be $s$, these simplify to $\frac{3s}{2}$ and $2s$, so the area of the larger polygon is $\frac{7s}{2}$. The dodceagon's area is $5s$, being the union of five squares with side length $s$ in the shape of a $+$ sign, so the area of the smaller region is $5s - \frac{7s}{2} = \frac{3s}{2}$. Then the ratio of the area of the larger region to the area of the smaller region is $\frac{7}{3}$, so $m+n = \boxed{10}$.

12. Nelson, who never studies for tests, takes several tests in his math class. Each test has a passing score of 60/100. Since Nelson's test average is at least 60/100, he manages to pass the class. If only nonnegative integer scores are attainable on each test, and Nelson gets a different score on every test, compute the largest possible ratio of tests failed to tests passed. Assume that for each test, Nelson either passes it or fails it, and the maximum possible score for each test is 100.

**Answer: 8**

**Solution:** If Nelson gets a score of 100/100 on one of the tests, he can be another 40 points behind the passing marks in total. Thus, he can earn scores of 59, 58, $\ldots$, down to 52, and then only has 4 deficit points left, so he cannot get a 51 on another test. If he does, he will need to pass another test (say, with a 99), but this lowers his ratio of tests failed to tests passed, because the 39 points earned from the 99 will need to be balanced by failures that carry 9 or more points of deficit; these cannot total to 8 or more tests. Hence, the maximum possible ratio is $\boxed{8}$.

13. For each positive integer $n$, let $f(n) = \frac{n}{n+1} + \frac{n+1}{n}$. Then $f(1) + f(2) + f(3) + \cdots + f(10)$ can be expressed as $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Compute $m+n$.

**Answer: 241**

**Solution:** Note that

$$f(n) = \left(\frac{n+1}{n+1} - \frac{1}{n+1}\right) + \left(\frac{n}{n} + \frac{1}{n}\right) = 2 + \frac{1}{n} - \frac{1}{n+1}.$$

So

$$\begin{aligned} f(1) + f(2) + f(3) + \cdots + f(10) &= 2 \cdot 10 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{10} - \frac{1}{11}\right) \\ &= 20 + \frac{10}{11} \\ &= \frac{230}{11}, \end{aligned}$$

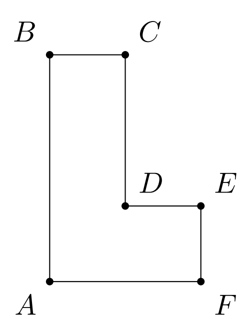
and $m + n = \boxed{241}$.

14. Triangle $\triangle ABC$ has point $D$ lying on line segment $\overline{BC}$ between $B$ and $C$ such that triangle $\triangle ABD$ is equilateral. If the area of triangle $\triangle ADC$ is $\frac{1}{4}$ the area of triangle $\triangle ABC$, then $\left(\frac{AC}{AB}\right)^2$ can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

**Answer: 22**

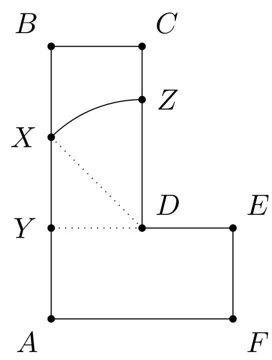
**Solution:** Let $AB = BD = AD = s$, and let the altitude of equilateral triangle $\triangle ABD$ from $A$ intersect $\overline{BD}$ at $E$. Then $BE = \frac{s}{2}$ and $AE = \frac{s\sqrt{3}}{2}$. Given that $\triangle ADC$ has area $\frac{1}{4}$ the area of $\triangle ABC$, we obtain $DC = \frac{s}{3}$, so $BC = \frac{4s}{3}$. Hence, $EC = \frac{5s}{6}$. By the Pythagorean Theorem, $AC^2 = AE^2 + EC^2 = \frac{3}{4}s^2 + \frac{25}{36}s^2 = \frac{13}{9}s^2$, so $AC = \frac{\sqrt{13}}{3}s = \frac{\sqrt{13}}{3}AB$, and $\frac{AC}{AB} = \frac{\sqrt{13}}{3}$, so $m + n = \boxed{22}$.

15. In hexagon $ABCDEF$, $AB = 60$, $AF = 40$, $EF = 20$, $DE = 20$, and each pair of adjacent edges are perpendicular to each other, as shown in the below diagram. The probability that a random point inside hexagon $ABCDEF$ is at least $20\sqrt{2}$ units away from point $D$ can be expressed in the form $\frac{a-b\pi}{c}$, where $a, b, c$ are positive integers such that $\gcd(a, b, c) = 1$. Compute $a + b + c$.




**Answer: 23**

**Solution:** Notice that $AD = DF = 20\sqrt{2}$ and that $20\sqrt{2} < CD$. With these two in mind, we can sketch out the points that are within $20\sqrt{2}$ units of $D$.




The area of the points inside the hexagon that are within $20\sqrt{2}$ units of $D$ is the area of rectangle $YEFA$, triangle $\triangle XYD$, and sector $XDZ$. The area of $YEFA$ is $40 \cdot 20 = 800$. By the Pythagorean Theorem, $XY = 20$, so the area of $XYD$ is $\frac{20 \cdot 20}{2} = 200$. Additionally, because $\triangle XYD$ is isosceles, $\angle YDX = 45^\circ$, so $\angle XDZ = 45^\circ$. Therefore, the area of sector $XDZ$ is $\frac{1}{8}\pi \cdot (20\sqrt{2})^2 = 100\pi$.

Thus, the area of the points inside the hexagon that are within $20\sqrt{2}$ units of $D$ is $800 + 200 + 100\pi = 1000 + 100\pi$. The area of the hexagon is $40 \cdot 20 + 20 \cdot 40 = 1600$. Thus, the probability that a point inside the hexagon is at least $20\sqrt{2}$ units from point $D$ is $\frac{600-100\pi}{1600} = \frac{6-\pi}{16}$. Then $a + b + c = \boxed{23}$.

16. The equation

$$\sqrt{x} + \sqrt{20 - x} = \sqrt{20 + 20x - x^2}$$

has 4 distinct real solutions, $x_1, x_2, x_3$, and $x_4$. Compute $x_1 + x_2 + x_3 + x_4$.

**Answer: 40**

**Solution:** The above equation can be rewritten as $\sqrt{x} + \sqrt{20 - x} = \sqrt{20 + x(20 - x)}$. Let $x = 20 - y$. We find that the equation becomes $\sqrt{20 - y} + \sqrt{y} = \sqrt{20 + (20 - y)y}$. Thus, $y$ is also a solution. This means that if $x$ is a solution to the equation, then $20 - x$ is also a solution. We can then assume without loss of generality that $x_3 = 20 - x_1$ and $x_4 = 20 - x_2$ to find that $x_1 + x_2 + x_3 + x_4 = 20 + 20 = \boxed{40}$.

17. How many distinct words with letters chosen from B, M, T have exactly 12 distinct permutations, given that the words can be of any length, and not all the letters need to be used? For example, the word BMMT has 12 permutations. Two words are still distinct even if one is a permutation of the other. For example, BMMT is distinct from TMMB.

**Answer: 108**

**Solution:** For a $m$-letter word, consider the number of copies of the most duplicated letter. If there are $m - 1$ copies, then we have $m$ permutations (as there are $m$ choices for the position of the unique letter). Thus, any 12-letter word with letters B, M, and T works; we have 3 choices for the letter that appears 11 times, and 2 choices for the unique letter, giving 6 choices for the number of each letter. For $6 \le m \le 11$, note that, for $m - 2$ copies of the most frequent letter, there must be at least $\binom{m}{2}$ permutations, in the event that both of the two remaining letters are also identical. However, $\binom{6}{2} = 15 > 12$, so this case produces no words. For $m = 5$, similarly, $\binom{5}{2} = 10$, but $\frac{5!}{3!1!1!} = 20$, skipping over 12. For $m = 4$, we have 3 choices for the duplicated letter, and must use the other 2 letters, giving 3 distinct choices for the number of each letter.

Finally, for $m < 4$, there are no possibilities, so in total, we have 9 choices for the number of each letter. Each word has 12 distinct permutations, so there are $12 \cdot 9 = \boxed{108}$ distinct words.

18. We call a positive integer *binary-okay* if at least half of the digits in its binary (base 2) representation are 1's, but no two 1s are consecutive. For example, $10_{10} = 1010_2$ and $5_{10} = 101_2$ are both binary-okay, but $16_{10} = 10000_2$ and $11_{10} = 1011_2$ are not. Compute the number of binary-okay positive integers less than or equal to 2020 (in base 10).

    **Answer: 21**

    **Solution:** Begin by noting that $2020_{10}$ has 11 digits in binary, as $2^{10} < 2020 < 2^{11}$. There are two cases: either the positive integer has an odd number of digits, or the positive integer has an even number of digits.

    Case 1: Odd number of digits:

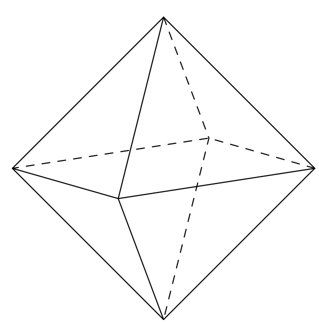
    If a positive integer has an odd number of binary digits, it must begin with a 1 and alternate between 0's and 1's, since it will necessarily have one more 1 than 0s. (If it were to have any more 1s, then by Pigeonhole Principle, at least two of them must be adjacent.)

    Case 2: Even number of digits:

    If it has an even number of binary digits, then exactly half of them must be 1s by a similar argument. Precisely, if it has $2n$ digits, then there are $n$ ways to arrange the 1s with no two consecutive. (If there are $n$ 1s, there must be $n-1$ 0s inserted between the 1s to prevent the 1s from being adjacent.) The final 0 can go anywhere except at the beginning, so there are $n$ positions for the final 0.

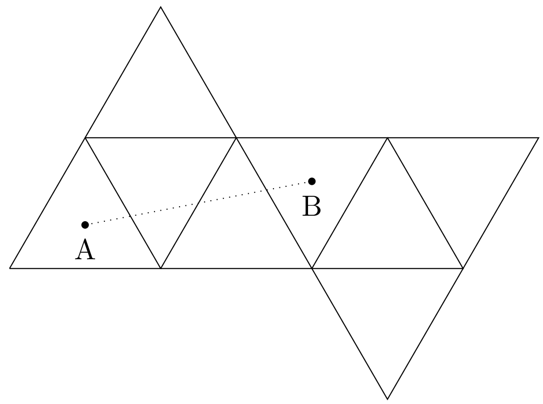
    Thus, there are $1+1+1+1+1+1 = 6$ binary-okay numbers with an odd number of digits (as $10101010101_2 < 1024 + 512 = 1536 < 2020$) and $5+4+3+2+1 = 15$ binary-okay numbers with an even number of digits. Altogether, we have $\boxed{21}$ binary-okay numbers under 2020.

19. A regular octahedron (a polyhedron with 8 equilateral triangles) has side length 2. An ant starts on the center of one face, and walks on the surface of the octahedron to the center of the opposite face in as short a path as possible. The square of the distance the ant travels can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m+n$.



    **Answer: 31**

    **Solution:**




In the unfolded net of the octahedron above, the distance the ant travels is the distance between $A$ and $B$. By 30-60-90 triangles, the height of each equilateral triangle is $\sqrt{3}$.
Then the vertical distance between $A$ and $B$ is $\frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{3}$ (since the center of an equilateral triangle is $\frac{1}{3}$ the way up of the altitude from the base). The horizontal distance between $A$ and $B$ is 3, so the distance between $A$ and $B$ is $\sqrt{3^2 + \left(\frac{\sqrt{3}}{3}\right)^2} = \sqrt{9 + \frac{1}{3}} = \frac{\sqrt{84}}{3}$. Then $AB^2 = \frac{84}{9} = \frac{28}{3}$, so $m + n = \boxed{31}$.

20. The sum of $\frac{1}{a}$ over all positive factors $a$ of the number 360 can be expressed in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

**Answer: 17**

**Solution:** Recall that if $a$ is a factor of 360, then there exists a number $d$ such that $ad = 360$, which implies that $d$ is **also** a factor of 360. We want to sum all terms of the form $\frac{1}{a}$, where $a$ is a factor of 360, which is equivalent to summing all terms of the form $\frac{1}{\frac{360}{d}} = \frac{d}{360}$, where $d$ is a factor of 360. Factoring, we have $360 = 2^3 \cdot 3^2 \cdot 5$, and by the sum of factors formula, the sum of all fractions of the form $\frac{d}{360}$ evaluates to

$$\frac{\left(2^0 + 2^1 + 2^2 + 2^3\right)\left(3^0 + 3^1 + 3^2\right)\left(5^0 + 5^1\right)}{360} = \frac{13}{4}.$$

Thus, $m + n = 13 + 4 = \boxed{17}$.