1. Frankie the frog likes to hop. On his first hop, he hops 1 meter. On each successive hop, he hops twice as far as he did on the previous hop. For example, on his second hop, he hops 2 meters, and on his third hop, he hops 4 meters. How many meters, in total, has he travelled after 6 hops?

   Answer: 63

   Solution: After 6 hops, he has hopped $1+2+4+8+16+32 = 2^6 - 1 = \boxed{63}$ meters.

2. Anton flips 5 fair coins. The probability that he gets an odd number of heads can be written in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

   Answer: 3

   Solution: Each sequence of coins with an odd number of heads corresponds to a sequence of coins with an even number of heads, constructed by swapping heads with tails and vice versa, so the probability is $\frac{1}{2}$. Thus, the answer is $\boxed{3}$.

3. April discovers that the quadratic polynomial $x^2 + 5x + 3$ has distinct roots a and b. She also discovers that the quadratic polynomial $x^2 + 7x + 4$ has distinct roots c and d. Compute

   $$ac + bc + bd + ad + a + b.$$

   Answer: 30

   Solution: Notice that

   $$\begin{aligned} ac + bc + bd + ad + a + b &= (a+b)c + (a+b)d + (a+b) \\ &= (a+b)(c+d) + (a+b) \\ &= (a+b)(c+d+1). \end{aligned}$$

   Now, we are given that

   $$x^2 + 5x + 3 = (x-a)(x-b) \qquad \text{and} \qquad x^2 + 7x + 4 = (x-c)(x-d),$$

   so $a+b = -5$ and $c+d = -7$. So our answer is $(a+b)(c+d+1) = (-5)(-6) = \boxed{30}$.

4. A rectangular picture frame that has a 2 inch border can exactly fit a 10 by 7 inch photo. What is the total area of the frame's border around the photo, in square inches?

   Answer: 84

   Solution: The photo and border together have a total area of $(10+2\cdot2)\cdot(7+2\cdot2) = 14\cdot11 = 154$ square inches. Because $10\cdot7 = 70$ of those square inches are occupied by the photo, the remaining area, which is left for the border, is $154 - 70 = \boxed{84}$ square inches.

5. Compute the median of the positive divisors of 9999.

   Answer: 100

   Solution: We note that $9999 = 99 \cdot 101$. Since 100 is not a divisor of 9999, we will have that half of the divisors are less than or equal to 99, and half of the divisors are greater than or equal to 101. So, the median of the divisors is $\frac{99+101}{2} = \boxed{100}$.

6. Kaity only eats bread, pizza, and salad for her meals. However, she will refuse to have salad if she had pizza for the meal right before. Given that she eats 3 meals a day (not necessarily distinct), in how many ways can we arrange her meals for the day?

Answer: 21

Solution: We work one meal at a time.

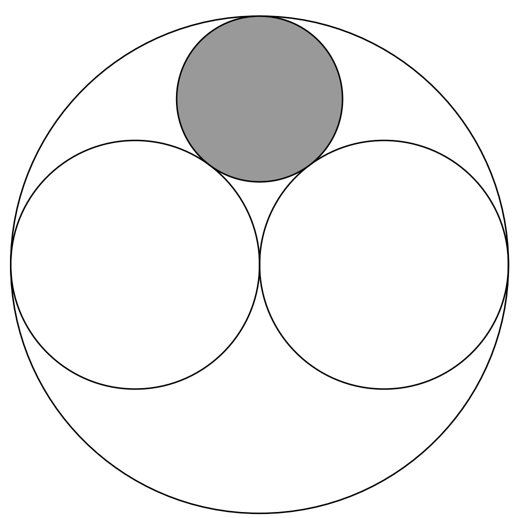
- If Kaity ate only one meal a day, then there would be 3 arrangements.
- If Kaity ate two meals a day, then there would be $3 \cdot 3 - 1 = 8$ arrangements, since pizza followed by a salad is not a valid arrangement.
- Consider the case where Kaity eats three meals a day. We can append each of the 3 items to the end of any valid two-meal arrangement. Some of the sequences that we just constructed are invalid, namely those that end in pizza then salad. There are 3 such sequences (corresponding to the one-meal arrangements for the first meal of the day). Hence, the number of valid three-meal arrangements is $3 \cdot 8 - 3 = \boxed{21}$.

7. A triangle has side lengths 3, 4, and x, and another triangle has side lengths 3, 4, and $2x$. Assuming both triangles have positive area, compute the number of possible integer values for x.

Answer: 2

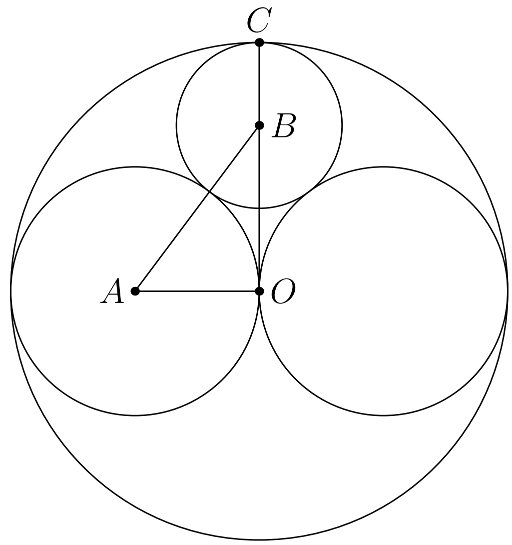
Solution: By Triangle Inequality for the first triangle, we need $x < 3 + 4 = 7$ and $x + 3 > 4$; the second is equivalent to $x > 1$. Similarly, for the second triangle, we need $2x < 3 + 4 = 7$ and $2x + 3 > 4$; again, the second is equivalent to $2x > 1$. Thus, $1 < x < \frac{7}{2}$, so $x = 2$ or $x = 3$, giving $\boxed{2}$ solutions.

8. In the diagram below, the largest circle has radius 30 and the other two white circles each have a radius of 15. Compute the radius of the shaded circle.



Answer: 10

Solution: Consider the following diagram.



We would like to solve for BC. We note that $\angle AOC$ is a right angle. Using the Pythagorean Theorem, we solve

$$\begin{aligned} 30 &= OC \\ 30 &= OB + BC \\ 30 &= \sqrt{AB^2 - AO^2} + BC \\ 30 &= \sqrt{(AO + BC)^2 - AO^2} + BC \\ (30 - BC)^2 &= (AO + BC)^2 - AO^2 \\ (30 - BC)^2 &= BC^2 + 30BC \\ 90BC &= 900 \\ BC &= 10. \end{aligned}$$

So, the radius of the shaded circle is $\boxed{10}$.

9. What is the remainder when 2022 is divided by 9?

   Answer: 6

   Solution: Note that summing the digits of a number does not change its remainder when dividing by 9. So, the remainder is $2 + 2 + 2 = \boxed{6}$.

10. For how many positive integers x less than 2022 is $x^3 - x^2 + x - 1$ prime?

    Answer: 1

    Solution: Observe that $x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1)$. If x is a positive integer greater than 2, then $x - 1$ and $x^2 + 1$ are positive integers greater than 1, so $x^3 - x^2 + x - 1$ has two nontrivial factors, and thus is composite and not prime. When $x = 2$,

$$\begin{aligned} x^3 - x^2 + x - 1 &= (x - 1)(x^2 + 1) \\ &= (1)(4 + 1) \\ &= 5 \end{aligned}$$

is prime. When $x = 1$,

$$\begin{aligned} x^3 - x^2 + x - 1 &= (x-1)(x^2+1) \\ &= (0)(0+1) \\ &= 0 \end{aligned}$$

is not prime. Thus, there is exactly $\boxed{1}$ positive integer x less than 2022 such that $x^3 - x^2 + x - 1$ is prime.

11. A sphere and cylinder have the same volume, and both have radius 10. The height of the cylinder can be written in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Answer: 43

Solution: We have that

$$\frac{4}{3} \cdot \pi \cdot 10^3 = \pi \cdot 10^2 \cdot h,$$

upon which solving gives $h = \frac{40}{3}$. So, the answer is $\boxed{43}$.

12. Amanda, Brianna, Chad, and Derrick are playing a game where they pass around a red flag. Two players "interact" whenever one passes the flag to the other. How many different ways can the flag be passed among the players such that

(1) each pair of players interacts exactly once, and

(2) Amanda both starts and ends the game with the flag?

Answer: 0

Solution: Suppose each pair of players interact exactly once. Then Amanda interacts exactly once with Brianna, exactly once with Chad, and exactly once with Derrick, so she interacts with other players an odd number of times (namely, 3).

Suppose Amanda starts and ends the game with the flag. Every time she interacts with another player, she either gives the flag to someone else, or someone else gives the flag to her. If she gives away the flag n times, then she must receive the flag n times. But this means that she interacts with the other players an even number of times.

Amanda must either interact with the other players an odd number of times or an even number of times—she cannot do both. It follows that there are $\boxed{0}$ games in which each pair of players interacts exactly once and Amanda both starts and ends the game with the flag.

13. Compute the value of

$$\frac{12}{1 + \frac{12}{1 + \frac{12}{1 + \ddots}}}.$$

Answer: 3

Solution: Let $x = \frac{12}{1+\frac{12}{1+\frac{12}{1+\cdots}}}$. Then we can write

$$x = \frac{12}{1+x}.$$

Solving for x gives $x^2 + x - 12 = 0$, which is $(x+4)(x-3) = 0$, so $x = 3$ or $x = -4$. Since the fraction is positive, we have an answer of $\boxed{3}$.

14. Compute the sum of all positive integers a such that $a^2 - 505$ is a perfect square.

Answer: 306

Solution: We are computing positive integer values (a, b) such that $a^2 - 505 = b^2$, which is equivalent to computing positive integer solutions to

$$(a+b)(a-b) = 505.$$

Without loss of generality, we take $b \geq 0$ so that $a + b \geq a - b$. The point is that, because $505 = 5 \cdot 101$, the only ways to factor 505 as a product of two integers are $505 \cdot 1$ or $101 \cdot 5$. In particular, because $a + b$ must be the larger factor, we are solving the two systems

$$\begin{cases} a+b=505 \\ a-b=1 \end{cases} \quad \text{and} \quad \begin{cases} a+b=101 \\ a-b=5. \end{cases}$$

These give $a = \frac{505+1}{2}$ and $a = \frac{101+5}{2}$ respectively, so the sum of all possible values of a is

$$\frac{1+101+5+505}{2} = \frac{102+510}{2} = \frac{612}{2} = \boxed{306},$$

which is what we wanted.

15. Alissa, Billy, Charles, Donovan, Eli, Faith, and Gerry each ask Sara a question. Sara must answer exactly 5 of them, and must choose an order in which to answer the questions. Furthermore, Sara must answer Alissa and Billy's questions. In how many ways can Sara complete this task?

Answer: 1200

Solution: First, Sara chooses who out of Charles, Donovan, Eli, Faith, and Gerry to answer. She must choose 3 people (so that she answers 5 when she also answers Alissa and Billy), so there are $\binom{5}{3}$ ways in which she can do this. Then, she must choose the order in which she will answer; there are $5!$ ways in which she can do this. By the product rule, there are $\binom{5}{3}5! = \boxed{1200}$ ways in which Sara can complete her task.

16. The integers $-x$, $x^2 - 1$, and x^3 form a non-decreasing arithmetic sequence (in that order). Compute the sum of all possible values of x^3.

Answer: 9

Solution: Since $-x$, $x^2 - 1$, and x^3 form an arithmetic sequence, we know that $2\left(x^2 - 1\right) = (x^3) + (-x)$. Simplifying, we find that

$$x^3 - 2x^2 - x + 2 = 0.$$

We see that we can factor $x - 2$ out of this to get $(x-2)\left(x^2-1\right) = (x-2)(x-1)(x+1)$. So our roots are $\{-1, 1, 2\}$. Plugging each root in, we obtain the following sequences:

- $x = -1$ gives $1, 0, -1$.
- $x = 1$ gives $-1, 0, 1$.
- $x = 2$ gives $-2, 3, 8$.

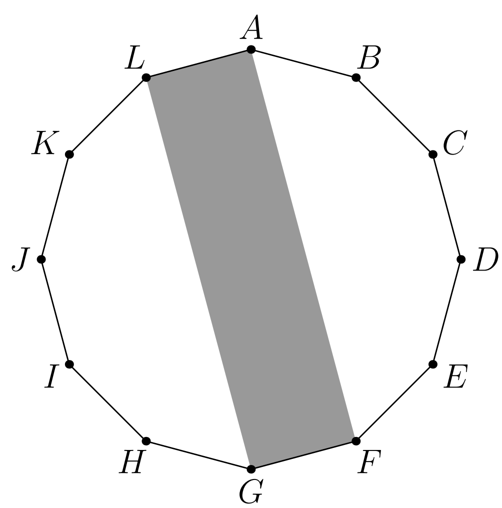
We see that the sequence generated by -1 is decreasing, but the other two sequences are increasing. Summing the possible values of x^3, we find that the answer is $1 + 8 = \boxed{9}$.

17. Moor and his 3 other friends are trying to split burgers equally, but they will have 2 left over. If they find another friend to split the burgers with, everyone can get an equal amount. What is the fewest number of burgers that Moor and his friends could have started with?

Answer: 10

Solution: Let k be the number of burgers that Moor and his three other friends would have gotten if they had tried to split the burgers equally. Then Moor would have started with $4k+2$ burgers. This needs to be a multiple of 5 in order for the number of burgers to finally split equally. The smallest k that makes this work is $k = 2$, giving $4 \cdot 2 + 2 = \boxed{10}$ burgers.

18. Consider regular dodecagon $ABCDEFGHIJKL$ below. The ratio of the area of rectangle $AFGL$ to the area of the dodecagon can be written in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.



Answer: 4

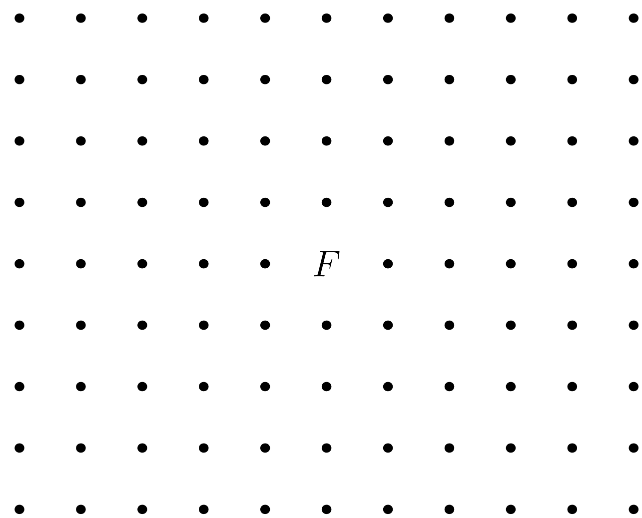
Solution: Draw diagonals $\overline{AG}$ and $\overline{FL}$, and call their intersection O. By radial symmetry, triangles $\triangle ALO$ and $\triangle GFO$ are each $\frac{1}{12}$ the area of the original dodecagon. Also, since the diagonals bisect each other, they split the rectangle into 4 triangles of equal area. Since one triangle is $\frac{1}{12}$ of the total area, four triangles would be $\frac{4}{12} = \frac{1}{3}$ of the total area. So, the answer is $1 + 3 = \boxed{4}$.

19. Compute the remainder when $3^{4^{5^6}}$ is divided by 4.

Answer: 1

Solution: Note that 4^n is even for positive integers n, and that 3^m gives a remainder of 1 when divided by 4 for positive even integers m. So, the remainder is $\boxed{1}$.

20. Fred is located at the middle of a 9 by 11 lattice (diagram below). At every second, he randomly moves to a neighboring point (left, right, up, or down), each with probability 1/4. The probability that he is back at the middle after exactly 4 seconds can be written in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.



Answer: 73

Solution: There are 3 cases:

- Fred moves only horizontally. In this case, he moves left twice and right twice. There are $\binom{4}{2} = 6$ ways to order this.
- Fred moves only vertically. This is symmetrical to the previous case, so there are also 6 ways to order this.
- Fred moves both horizontally and vertically. In this case, he moves left, right, up, and down once. There are $4! = 24$ ways to order this.

Adding all of these, there are $24+6+6 = 36$ ways to walk such that Fred is back at the middle. Since each step is done with probability $1/4$, the probability of a certain path will be $\frac{1}{4^4} = \frac{1}{256}$. Therefore, the probability that he is back at the middle is $\frac{36}{256} = \frac{9}{64}$. So, the answer is $\boxed{73}$.