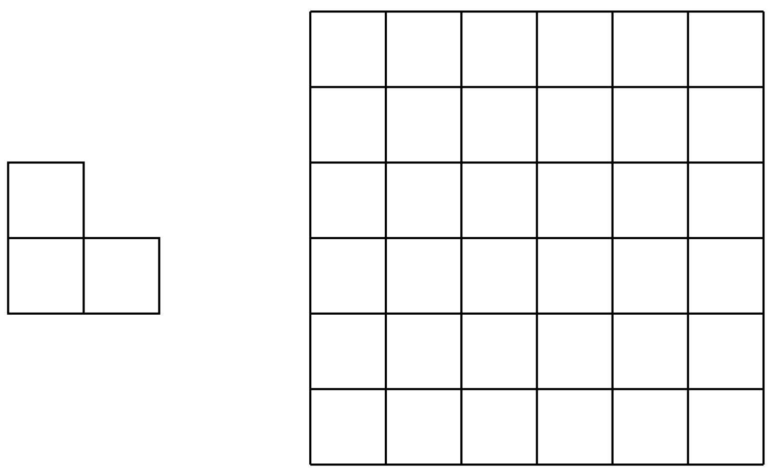
Time limit: 40 minutes.
Instructions: This test contains 6 sets of questions, with 20 questions in total. You may work on any problem from any set at any time. Submit a single answer sheet for grading. Only answers written inside the boxes on the answer sheet will be considered for grading.
Note: Some questions may depend on other questions' answers. The drawing at the top of every set indicates which other answers are used for which problems. All answers for this round are numerical values (not in terms of any variables).
No calculators.

Set 1

$$1 \longrightarrow 2$$

1. What is the maximum number of L-shaped pieces (the piece on the left) that can fit in the 6×6 grid shown below? You may assume that you can rotate the L-shaped pieces.



2. Let N_1 be the answer to question 1.

   Suppose $N_1 + 1 + \frac{1}{N_1}$ can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute a.

Set 2

$$3 \longrightarrow 4 \longrightarrow 5$$

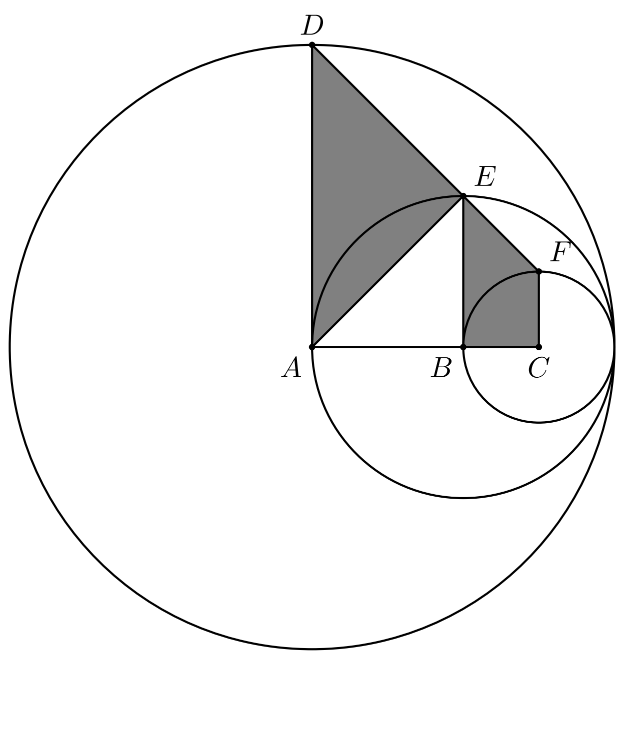
3. Every day after school, Alice drops 1 marble on the ground, Bob drops 2 marbles on the ground, and Carol drops 3 marbles on the ground. After some number of days, there are a total of 24 marbles on the ground. How many of these marbles were dropped by Bob?

4. Let N_3 be the answer to question 3.

   The hype-house for Beta Mu Tau has N_3 doors, each numbered 1 through N_3. Jingyuan enters through a door and exits through a door (which doesn't have to be a different door from the first), such that the sum of the numbers on the two doors is even. Compute the number of possible ways Jingyuan could enter and exit the hype-house.

5. Let N_4 be the answer to question 4.

   Three circles with centers A, B, and C are internally tangent to each other at a common point, as shown in the diagram below, and have radii N_4, $\frac{N_4}{2}$, and $\frac{N_4}{4}$, respectively. Points D, E, and F lie on the circles centered at A, B, and C, respectively, such that angles $\angle DAB = \angle EBC = \angle FCB = 90°$. What is the area of the shaded figure?



Set 3

$$6 \longrightarrow 7 \longrightarrow 8 \longrightarrow 9$$

6. Points A, B, and C lie on a line, in that order, from left to right. Point D is plotted such that triangle $\triangle ADB$ is equilateral. If $BC = 2$ and $DC = 2\sqrt{13}$, compute AC.

7. Let N_6 be the answer to question 6.

   Shreyas plays a game with N_6 buckets. During a round, he chooses two buckets uniformly at random to keep track of, then drops a ball into one of the N_6 buckets uniformly at random. If the ball lands in one of the chosen buckets, the game ends. Otherwise, he removes the ball and the bucket that the ball landed in, and plays another round. Compute the probability that this game will continue until Shreyas has two buckets left.

8. The answer to question 7 can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Let N_7 be $a + b$.

   There are N_7 red dots and some number of blue dots connected by line segments. Each red dot is connected to exactly 4 blue dots, and each blue dot is connected to exactly 2 red dots. Furthermore, each blue dot is connected to exactly 2 other blue dots, and no two red dots are connected. Compute the number of line segments drawn.

9. Let N_8 be the **sum of the digits** of the answer to question 8.

   For each positive integer k, define the integers

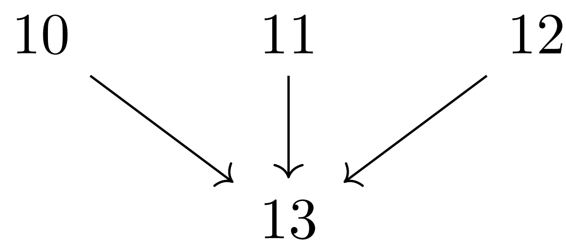
   $$a_k = k + 2k + \cdots + (k-1)k + k \cdot k$$

   and

   $$b_k = (k)(2k)\cdots((k-1)k)(k \cdot k).$$

   What is the largest positive integer k less than or equal to N_8 such that b_k is NOT divisible by a_k?

Set 4



10. Albert has two cups, A and B, containing a and b liters of water, respectively. Albert pours one-third of cup A's contents into cup B, and then pours one-half of cup B's contents into cup A. He notices that cups A and B now contain b and a liters of water, respectively. If the water in both cups combines for a total of 24 liters, compute a.

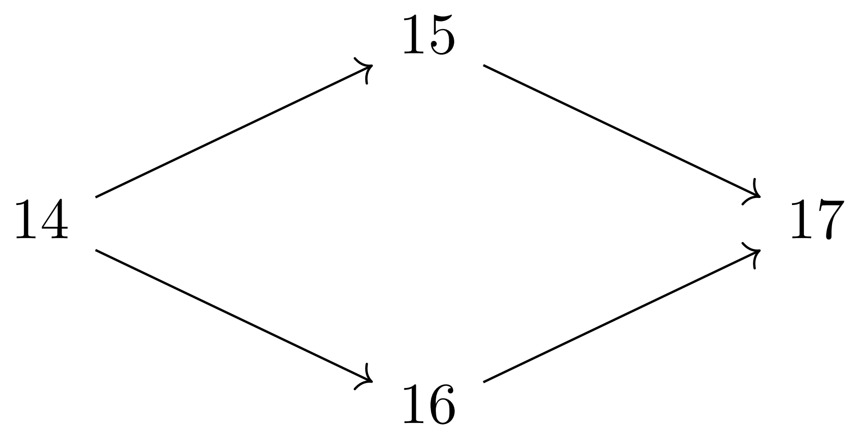
11. Compute the 4-digit perfect square of the form $\underline{AABB}$, where A and B are digits and $A \neq 0$.

12. The value 56 is written in its binary form as 111000. Jingyuan takes all rearrangements of these ones and zeroes (including 111000) and sets the leftmost digit of each rearrangement equal to 0 (regardless if the digit was a 1 or 0 before). He then sums up all the resulting binary values, and then converts this sum to base 10. Compute this sum.

13. Let N_{10} be the answer to question 10, N_{11} be the **sum of the digits** of the answer to question 11, and N_{12} be the **sum of the digits** of the answer to question 12.

    Two circles O_1 and O_2 are externally tangent to line ℓ at the same point A, and the two circles are externally tangent to each other. Point B lies on line ℓ such that $AB = N_{11}$. There is a line through point B that intersects circle O_1 at points D and E, where $BD < BE$ and $BD = N_{10}$, and another line through point B that intersects circle O_2 at points F and G, where $BF < BG$ and $BF = N_{12}$. Compute $\frac{DG}{EF}$.

Set 5



14. Anton has two standard six-sided dice where opposite faces always add up to 7. He puts the first die on a table, and stacks the second die on top of the first die so that there are 9 visible faces and 3 non-visible faces. The product of the values of the non-visible faces does not divide the product of the values of the visible faces. What is the sum of the non-visible faces?

15. Let N_{14} be the answer to question 14.

    Compute the area in the xy-plane above the line $y = 0$ and to the left of the line $x = N_{14}$, but below the graph of $y = N_{14}\lfloor x \rfloor$. (Here, $\lfloor m \rfloor$ is defined as the greatest integer less than or equal to m. For example, $\lfloor 3.14 \rfloor = 3$ and $\lfloor 4 \rfloor = 4$.)

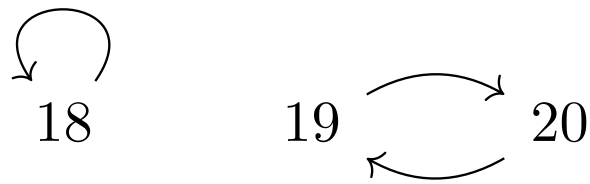
16. Let N_{14} be the answer to question 14.

    How many sequences of distinct positive integers starting with 1 and ending with $2^{N_{14}}$ are there such that each integer of the sequence (except for 1) is divisible by the previous integer?

17. Let N_{15} be the **sum of the digits** of the answer to question 15 and N_{16} be the **sum of the digits** of the answer to question 16.

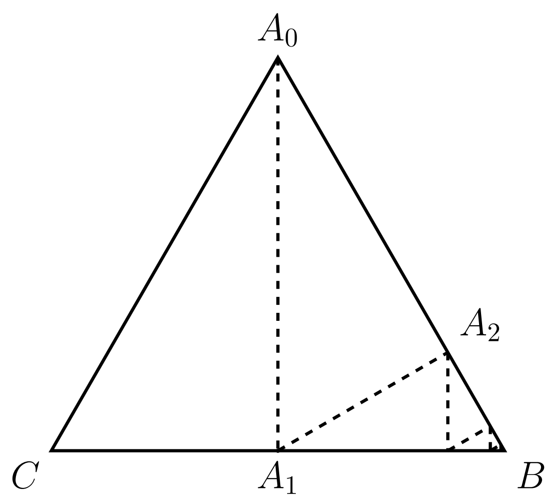
    Consider two circles D_O and D_P centered at points O and P with radii N_{15} and N_{16} respectively, and suppose they intersect at points W and X with $\overline{OX} \perp \overline{PX}$. Let ℓ be a line passing through X which intersects circle D_O elsewhere at A and circle D_P elsewhere at B. Compute the maximum possible value of AB.

Set 6



18. Let N_{18} be the answer to this question.

    The equilateral triangle $\triangle A_0BC$, as shown in the diagram below, has side length $\sqrt{3}(N_{18}-2)$. Let the altitude from A_0 intersect $\overline{BC}$ at A_1. Then for each positive integer i, an altitude from A_i intersects $\overline{A_{i-1}B}$ at A_{i+1}. This process is continued indefinitely, and points A_0, A_1, and A_2 are shown in the diagram. Compute the sum of the lengths $A_0A_1 + A_1A_2 + A_2A_3 + \cdots$ (these lengths are represented by the dashed lines in the diagram).



19. Let N_{20} be the answer to question 20.

    Leanne builds a pyramid out of equally sized blocks. The base has N_{20} blocks, and each successive layer has 2 fewer blocks. If there is exactly 1 block on the top level of the pyramid, how many blocks are in the pyramid?

20. Let N_{19} be the answer to question 19.

    The integer $N_{19}-1$ can be written in the form p^2q^3, where p and q are distinct one-digit prime numbers. Compute $10p+q-2$.