Introduction

The Billiards Masters Tournament (BMT), in an effort to boost its reputation, is about to host its annual crazy billiards tournament and is hiring you to analyze its newest tables. Your job is to determine crazy mathematical properties that BMT can use to help its billiard masters get better at their trade.

Good luck, and have fun!

Mathematical Billiards

The mathematics of Billiards is essentially the same as the mathematics that physicists use when examining optics. In general, when light hits a mirror, the trajectory of light obeys the **law of reflection**, explained in the figure below. Similarly, when a billiards ball hits the edge of the table, it always bounces off according to this law. This technique has been employed by Billiards Masters for years.

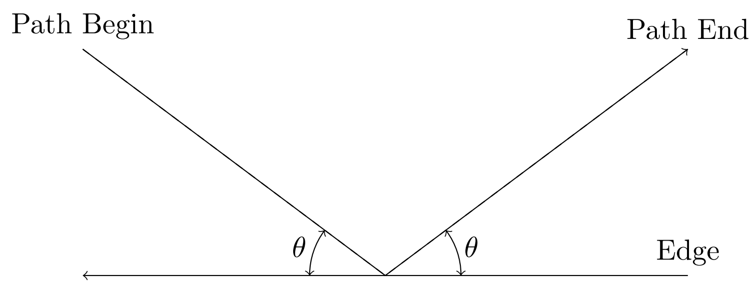


Figure 1: The Law of Reflection: Trajectories once they hit an edge, bounce off of that edge with the same angle they entered at.

A **billiards table** T is a two dimensional bounded polygon with special points called **pockets**. We note that the corners of T are *always* pockets. A curve on T is called a **trajectory** if it travels along a straight line, follows the law of reflection when it hits the edge of the table, and stops when it hits a pocket. The reason for this is that the law of reflection is undefined at corners.

We can think of a trajectory as a function of time $P(t)$ that takes in a non-negative time t and returns a position of a ball on the billiards table after t seconds, assuming that all billiard balls move at a speed of one unit per second.

Since the law of reflection is symmetric, we can also imagine "running the clock in reverse." So we define $P(t)$ for $t < 0$ as the trajectory $\overline{P}(-t)$, where $\overline{P}(t)$ is the trajectory of the ball starting at $P(0)$ and moving in the opposite direction as P.

The **starting point** of the ball is therefore $P(0)$. Starting points can be anywhere on the table, including on the edges, as long as they are not pockets.

Analyzing Billiard Trajectories

Definition. We say a billiards trajectory $P(t)$ on a surface is **periodic** if it "loops" back onto itself. That is, there exists a positive constant C such that $P(t+C) = P(t)$ for all t. The minimum positive constant C for which this is true is known as the **period** of the trajectory.

Definition. A trajectory P is **degenerate** if there exists a $t > 0$ such that $P(t)$ is a pocket.

Definition. A trajectory $P_1(t)$ **contains** a trajectory $P_2(t)$ if there exists a positive constant C such that $P_1(t + C) = P_2(t)$. We call P_1 a **rewind** of P_2. Similarly, P_2 is a **fast-forward** of P_1.

Definition. The **combinatorial period** of a periodic trajectory T on a mathematical billiards table is the size of the following set:

$$\{0 < t \leq C \mid P(t) \text{ is on an edge}\}$$

where C is the period of T. In other words, it is the number of times the trajectory hits an edge before repetition.

Note: The trajectory can hit the same edge multiple times, and in fact it can hit the same point multiple times. These are counted as distinct.

Definition. The **slope** of a trajectory is the initial slope of the the line $\overline{P(0)P(t_1)}$ where t_1 is the first time P hits a wall.

Definition. An n**-billiards table** is a regular n-gon with unit side lengths and pockets at all corners. We also implicitly embed the n-billiards table on the 2D-plane such that it is oriented such that its bottom-most edge, called the **base**, is on the positive x-axis, and the left-most point of the base is the origin.

Example. Here is an example of a periodic billiard trajectory on the 4-billiards table (i.e unit square now treated as a billiard table).

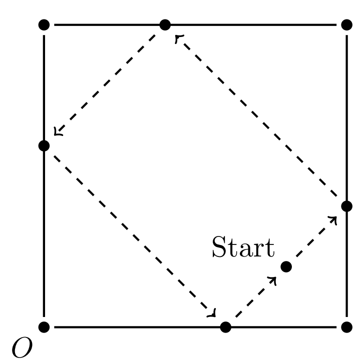


Figure 2: A periodic trajectory (dashed) on the unit square. Starts at $(0.8, 0.2)$ moving at a $45°$ angle with the base.

Remark. For the entire power round, whenever angles are specified, they are specified as **counter-clockwise** from the base of the billiards table. For instance, in the figure above, the periodic trajectory is moving at a 45 degree angle from the base of the 4-billiards table.

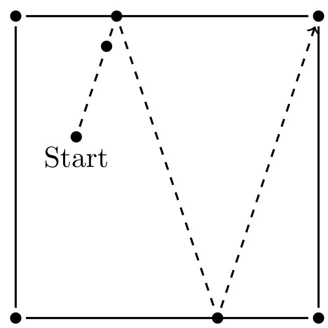
1. Draw out the following trajectories and determine whether the following trajectories are either periodic or degenerate on the 4-billiards table. If they are periodic, determine their combinatorial period. No proof required.

   (a) **[5]** A trajectory that starts at $(0.2, 0.6)$ moving towards the point $(0.3, 0.9)$.

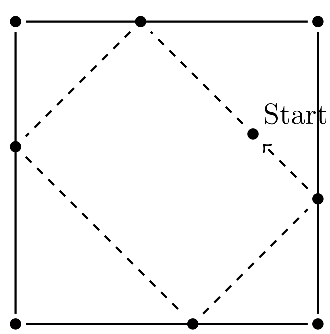
   (b) **[5]** A trajectory that starts at $\left(\frac{\pi}{4}, \frac{\pi}{5}\right)$ moving diagonally up and to the left (i.e. at a 135° angle relative to the base).

Solution to Problem 1:

(a) Extending the trajectory, we see that it will intersect the top edge at $\left(\frac{1}{3}, 1\right)$. The next time is at $\left(\frac{2}{3}, 0\right)$ and after that it hits $(1, 0)$, which is a degenerate point. Therefore, the trajectory is $\boxed{\textbf{degenerate}}$.



(b) Extending the trajectory, we see that this forms a rectangle within the larger square, since the intersection angles happen at 45° . None of these points hit on the edge are integral, so therefore the path repeats and it is $\boxed{\textbf{perodic}}$ with combinatorial period 4.



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2. **[10]** Show that for any periodic trajectory on any billiards table, its combinatorial period must be greater than 1.

Solution to Problem 2:

Extend the path in both positive and negative time. Since the path is contained within a bounded billiards table, both sides must eventually hit an edge. This point cannot be the same, so the path must hit at least 2 edges, so the period is at least 2.

3. (a) **[3]** Show that if trajectory P_1 contains P_2 then P_1 is degenerate if and only if P_2 is degenerate.

(b) **[4]** Show that if trajectory P_1 contains P_2 then P_1 is periodic if and only if P_2 is periodic.

(c) **[3]** Show that every periodic trajectory P has a rewind Q such that $Q(0)$ is on the edge of the table.

Solution to Problem 3:

(a) If P_1 is degenerate then there exists a C such that $P_1(C)$ is a degenerate point. Since P_1 contains P_2 there must exist a $C' < C$ by necessity such that $P_1(t+C') = P_2(t)$. But then $P_1(C) = P_2(C-C')$ is a degenerate point. Since $C'-C > 0$, we see that this value exists and so P_2 is degenerate.

Conversely, if P_2 is degenerate then there exists C_1 such that $P_2(C_1)$ is a degenerate point. Since P_1 contains P_2 there exists a C_2 such that $P_1(t+C_2) = P_2(t)$. Consequently $P_1(C_1 + C_2) = P_2(C_2)$ which is a degenerate point, so P_1 is degenerate.

(b) If P_1 is periodic then there exists a C such that $P_1(C + t) = P_1(t)$ for all t. Since P_1 contains P_2 there must exist a C' such that $P_1(t + C') = P_2(t)$. But then $P_2(t) = P_1(t + C') = P_1(t + C + C') = P_2(t + C')$ for all t. Therefore P_2 is periodic.

Conversely let P_2 be periodic, so $P_2(t + C) = P_2(t)$ for some C. Then by the definition of rewind we see that there exists C_1 such that $P_2(t - C_1) = P_1(t)$ for all t. Therefore $P_1(t) = P_2(t - C_1) = P_2(t + C - C_1) = P_1(t + C)$ for all t and we are done.

(c) If P is periodic, let t' be such that $P(t')$ is the point on the last edge that P hits before repeating. Let $Q(t) = P(t + t')$. Then, $Q(t)$ is a fast-forward of $P(t)$. Additionally, by periodicity of Q, $Q(t + C) = P(t + t')$. Since $C > t'$, we see that $Q(t + C - t') = P(t)$ so Q is a rewind of P with $Q(0)$ on the edge of the table.

Periodic Trajectories on Regular Polygons

(a) The 3-billiards table (b) The 5-billiards table

Figure 3: The black dots represent pockets and origin is located at the bottome left pocket for each table.

4. (a) **[1]** Draw the periodic trajectory on the 4-billiards table starting at the point $(\frac{1}{2}, 0)$ and moving at an angle of 45° from the base. Compute its combinatorial period?

(b) **[1]** Draw the periodic trajectory on the 3-billiards table starting at the point $(\frac{1}{2}, 0)$ and moving at an angle of 60° from the base. Compute its combinatorial period?

(c) **[1]** Draw the periodic trajectory on the 5-billiards table starting at the point $(\frac{1}{2}, 0)$ and moving at an angle of 72° from the base. Compute its combinatorial period?

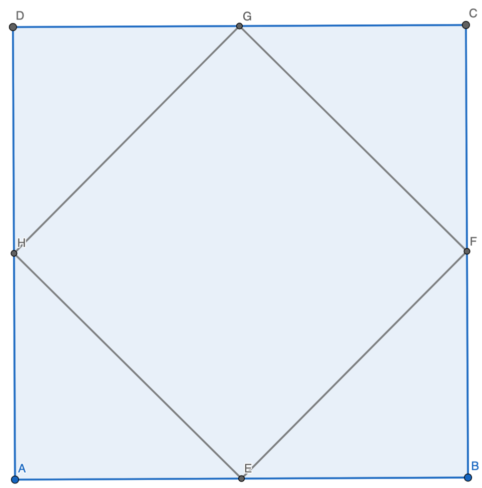
(d) **[4]** Consider an n-billiards table. Label the pockets of the table $C_0 \dots C_{n-1}$ counter-clockwise from the origin and let the points $P_0 \dots P_{n-1}$ be defined such that P_i is the midpoint of $\overline{C_i C}_{i+1 \pmod n}$. Show that for $i \neq j$, $\angle C_{i+1 \pmod n} P_i P_j$ is $\left(\frac{180(j-i \pmod n))}{n}\right)^{\circ}$.

Note: $C_i C_{i+1 \pmod n}$ means that $i, i+1$ are taken modulo n (i.e $C_0 = C_n$).

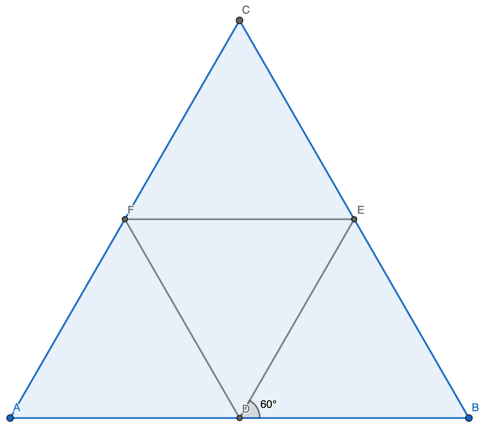
(e) **[3]** Show that for any n-billiards table, a trajectory starting at the midpoint of the base and traveling at an angle of $\left(\frac{180k}{n}\right)^\circ$ from the base is periodic, where $0 < k < n$ and k is an integer. Determine, with proof, its combinatorial period in terms of k and n?

Solution to Problem 4:

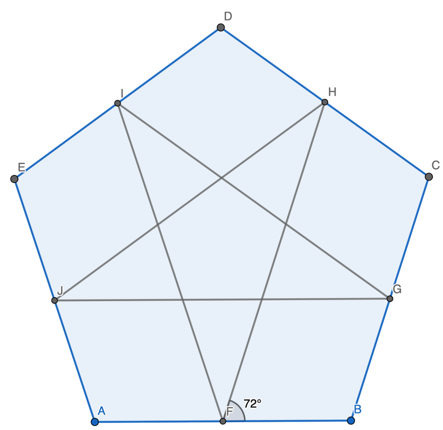
(a) The combinatorial period is $\boxed{4}$.



(b) The combinatorial period is $\boxed{3}$.



(c) The combinatorial period is $\boxed{5}$.



(d) By symmetry of the table we can rotate P_i such that it is the same as P_0. Thus, WLOG, $i = 0$, and we need only prove that $\angle C_1P_0P_j = \frac{180j}{n}^\circ$. Inscribe a circle inside the polygon centered at O. Then, the arc-lengths of O subtended by P_0P_j have the same arc measure, namely $\frac{360j}{n}^\circ$. Now, by the inscribed angle theorem in circles, $\angle C_1P_1P_j$ is exactly half the arc-length from P_1 to $P_j = \frac{360k}{n}^\circ$. So $\angle C_1P_1P_j = \frac{180j}{n}^\circ$.

(e) By part d above, P_0P_k is the first segment in the trajectory. By symmetry, the angle of incidence of this line segment at P_k is also equal to $\left(\frac{180k}{n}\right)^\circ$, so by the law of reflection, the angle of reflection off of P_k is also $\left(\frac{180k}{n}\right)^\circ$. Therefore, by another application of part d, the path hits P_{2k}. This repeats until we hit P_0 again. Therefore, we see we need to find the minimum c such that $ck \equiv 0 \pmod{n}$. This value always exists (for instance $c = n$ will always work). Therefore the path is periodic. The combinatorial period is simply the minimum value of c. Since c is minimal and k is fixed, and $n|ck$, we see that ck must be $\text{lcm}(k, n)$, since it is the lowest possible value that is a multiple of both k and n. Therefore c is simply $\boxed{\frac{\text{lcm}(n,k)}{k}}$, and we are done. Another answer which is equivalent is $\boxed{\dfrac{n}{\gcd(n,k)}}$.

5. (a) **[1]** Draw out the periodic trajectory on the 3-billiards table starting at the point $\left(\frac{1}{2}, 0\right)$ and moving at an angle of 30° from the base.

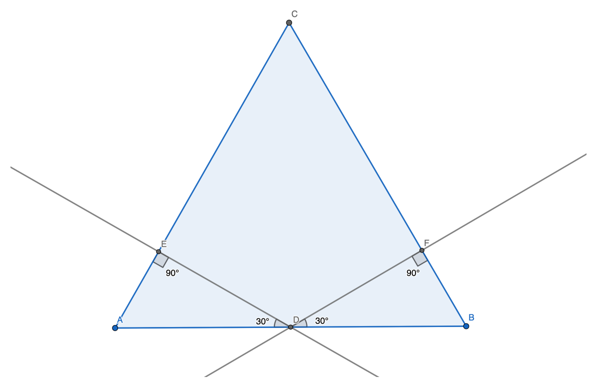
(b) **[1]** Draw out the periodic trajectory on the 5-billiards table starting at the point $\left(\frac{1}{2}, 0\right)$ and moving at an angle of 54° from the base.

(c) **[6]** Consider an n-billiards table T with n odd. Suppose we label the vertices of T with $A_1, A_2, \ldots, A_n$ starting at the topmost vertex moving clockwise. Let P be the midpoint of the base. Show that there exists a point Q on $\overline{A_1A_2}$ such that $\angle PQA_1$ is a right angle.

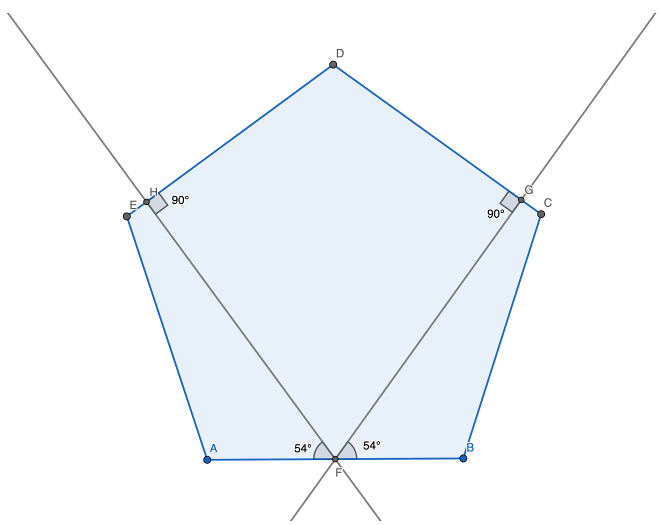
(d) **[2]** Show there exists a periodic trajectory of combinatorial period 4 on any n-billiards table with n odd.

Solution to Problem 5:

(a) This trajectory has combinatorial period $\boxed{4}$

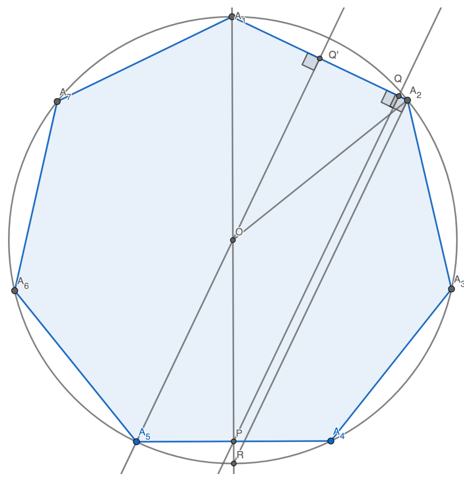


(b) This trajectory has combinatorial period $\boxed{4}$



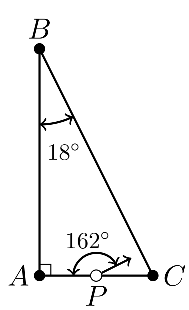
(c) **Solution 1:** Inscribe the polygon in a circle. Let O be the center of that circle. Then $\angle A_1OA_2$ is $\frac{360}{n}^\circ$ and the triangle $\triangle OA_1A_2$ is isosceles. Drop an altitude from O to A_1A_2 called Q'. Now, extend line A_1O through point P to point R on the other side of the circle. Then, $\triangle RA_2A_1$ is a right angle and $\triangle RA_2A_1$ and $\triangle OQ'A_1$ are similar. Now, we can move point O towards point R and grow triangle $\triangle OQ'A_1$ into triangle $\triangle RA_2A_1$. At some point this growth will hit point P, which will have a corresponding point Q on the segment $Q'A_2$. By similarity, this will yield a point Q such that PQA_1 is a right angle.

Here is an example of the phenomenon with a regular heptagon:



(d) Let the special point from the previous problem be called Q. Now, take the trajectory starting at the midpoint of the base to Q. Since the angle is exactly 90°, after hitting the opposing edge, the trajectory will reflect right back to Q at some angle θ. Since the polygon is regular, it has reflectional symmetry. Therefore, once hitting Q, the same exact path will happen–this time reflected–on the other half of the polygon. Once it returns, this cycle will repeat. The path hit a total of 4 edges before repetition, so this is a periodic path of period 4.

6. Consider billiards table formed by the right triangle $\triangle ABC$ with pockets at its corners below (not to scale). The base is $\overline{AC}$.

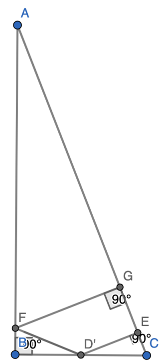


(a) **[4]** Draw the periodic trajectory starting at midpoint P of $\overline{AC}$ and moving at an 18° angle relative to segment $\overline{AC}$.

(b) **[6]** Let T be a table constructed from any right triangle $\triangle ABC$ with pockets at its corners. Show that T has a periodic trajectory with combinatorial period 6.

Solution to Problem 6:

(a) The combinatorial period is $\boxed{6}$.



(b) Define M as the midpoint of the **shortest** leg of the right triangle across an interior angle α. WLOG we will assume this leg to be $\overline{AC}$. Orient the right triangle as seen in the diagram such that the shortest leg is the base of the triangle and the longest leg $\overline{AB}$ is on its left.

Consider point D such that D is on $\overline{BC}$ and $\angle CMD = \alpha$. Similarly, consider point E on $\overline{AD}$ such that $\angle AME = \alpha$. Finally, consider point F on $\overline{BC}$ such that $\angle EFC = 90°$. Now, by some angle chasing, we see that $\angle PDC = 90°$, and $\angle PEA = \angle FEB = 90° - \alpha$. Thus the path moving from P to D to P to E to F to E to P follows the law of reflection and is periodic with combinatorial period 6.

Note: we need to check that E is actually on segment $\overline{AB}$, not just its line extension. Since $\overline{AC}$ is the shortest side, we see that α is the smallest angle. Thus, since $\triangle AME \sim \triangle ABC$, $\underline{AE}$ is smaller than AM. So $AE < AM < AC < AB$, and we conclude E is on segment $\overline{AB}$.

You might be wondering why we need to show this: try to think of what happens if M is the midpoint on the longest side of the triangle instead.

7. This problem is an extension of problem 4. We **highly** recommend doing it before attempting this one.

(a) **[1]** Draw the following trajectory: a periodic trajectory on the 3-billiards table starting at the point $(\frac{3}{4}, 0)$ and moving at an angle of 60°. Compute its combinatorial period?

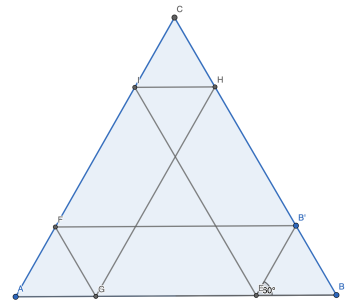
(b) **[1]** Draw the following trajectory: a periodic trajectory on the 5-billiards table starting at the point $(\frac{3}{5}, 0)$ and moving at an angle of 72°. Compute its combinatorial period?

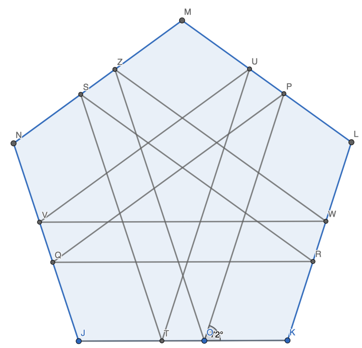
(c) **[6]** Consider an n-billiards table. Show that for any point p on the base (not including pockets), any non-degenerate trajectory starting from p moving at an angle of $\left(\frac{180k}{n}\right)^\circ$ is periodic, where $0 < k < n$ and k is an integer.

(d) **[2]** Consider an n-billiards table. Show that for any point p in the **interior of the table** (not including edges or pockets), any non-degenerate trajectory starting from p moving at an angle of $\left(\frac{180k}{n}\right)^\circ$ from the base is periodic, where k is any integer.

Solution to Problem 7:

(a) This has combinatorial period $\boxed{6}$





(b) This has combinatorial period $\boxed{10}$

(c) We know that by problem 4, if p is the midpoint of the base, this is true. Let T be the periodic trajectory from the midpoint of the base with the same angle. Let us examine what happens as we move M towards p, while maintaining the initial direction of the trajectory. For notation, let P be the trajectory of p.

Let $T_0 = M$ and T_1 be the first two points on the periodic trajectory T with T_0 on side $\overline{A_0B_0}$ and T_1 a midpoint on side $\overline{A_1B_1}$ such that A_i is to the left of B_i. Since $\angle T_0T_1A_1 = \angle T_1T_0A_0$, we see that $\overline{T_0T_1}||\overline{A_0A_1}$ and similarly with $\overline{B_0B_1}$. Thus, moving T_0 towards A_0 while maintaining the direction of the trajectory moves T_1 towards A_1 and similarly with $\overline{B_0B_1}$. Therefore, for any point P_0 on $\overline{A_0B_0}$ that has a trajectory aligned with T, its second point P_1 is on $\overline{A_1B_1}$ and $\overline{P_0P_1}$ is parallel to $\overline{T_0T_1}$.

Let $\overline{P_0P_1}$ where $P_0 = P$ be the first segment of the trajectory of P. Then $\overline{P_0P_1}$ by the logic above is parallel to $\overline{T_0T_1}$ and P_1 is on A_1B_1. As such, it reflects off of $\overline{A_1B_1}$ at point P_1 in a trajectory that must be parallel to $\overline{T_1T_2}$. Continuing like this, we see that the points $P_0 \dots P_n$ hit the same sides in the same order as $T_0 \dots T_n$ for any n.

Now, let d_i be the distance from P_i to the clockwise vertex on its edge (that is, the vertex / pocket the P_i's side on the billiards table going clockwise). Looking at $\overline{P_0P_1}$ we see that it is parallel to $\overline{A_0A_1}$ and, by symmetry it in fact forms an isosceles trapezoid. However, we see that since A_0 is to the left of P_0 which is on the base, $\overline{A_0P_0} = d_0$. But since A_1 is to the left of P_1, it is not clockwise from P_1, and thus $\overline{A_1P_1} = 1 - d_1$. Since the two distances are equal (by the properties of an isosceles trapezoid), $d_1 = 1 - d_0$. Repeating this for all edges in the periodic path, we see that the distances alternate.

Now, suppose T has a combinatorial period k that is even. Then P hits the same sides as T and the distances alternate with period 2. Thus, since k is even, after k reflections, we reach a point Q that is on the same edge as p and that is the same distance away from its clockwise vertex as p, and so $Q = p$ and the path is periodic with period k.

Now if k is odd, then after k reflections, we reach a point Q that is on the same size as p but instead with distance $1 - d_0$. However after $2k$ reflections, we reach a point Q' on the same side as P with the same distance away from its clockwise vertex as p. Thus $Q' = p$ and so P is periodic with period $2k$.

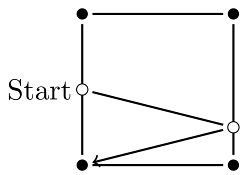
Thus, the path P is periodic.

(d) Since the trajectory is non-degenerate, let us rewind until it starts on the table at point Q. Now, rotate the table such that Q is on the base. Since p was traveling at an angle $\left(\frac{180k}{n}\right)^\circ$ with the original base, and we rotated the table by an angle of $\left(\frac{360p}{n}\right)^\circ$ where p is some integer. The rewinded trajectory is still traveling at an angle of $\left(\frac{180k}{n}\right)^\circ$ with the new base. However, in this case we know that $0 < k < n$, since the trajectory would have been horizontal had $k = 0$ or $k = n$, implying that p is not on the interior of the table. Thus, we may apply part c to see that the trajectory is periodic.

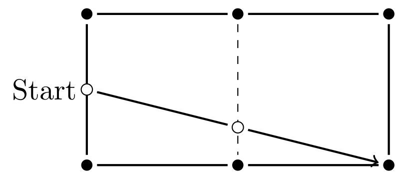
Unfolding Billiards Tables

One way to analyze trajectories on Billiard tables is through a technique called **unfolding**. Let us illustrate this phenomenon with a simple example. Consider the 4-billiards table trajectory that starts on the midpoint of the left edge reflects off of the right most edge and enters the bottom left pocket. If we were to reflect the shape (or unfold the shape) over the right-most edge, the trajectory would become a straight line!

Example. We will show how to unfold a square billiards table into larger rectangular one.



The starting Billiards Table



After reflection across the rightmost edge.

This holds true for any billiards table! In general, once you unfold a billiards table, it becomes another billiards table, where the edge that was reflected over disappears. Similarly, unfolding a periodic trajectory always yields another periodic trajectory and vice versa. We can also unfold multiple times over before deleting the edges.

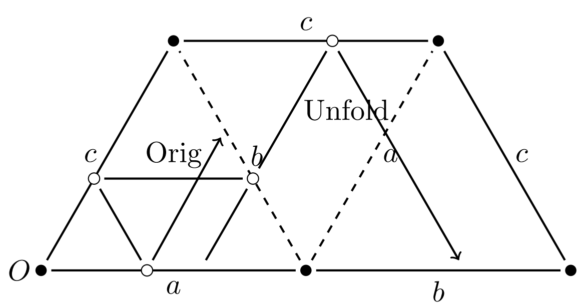
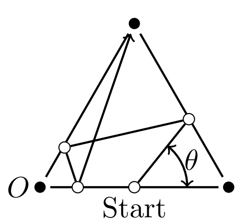


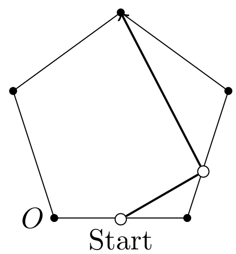
Figure 5: First unfolding over edge b to create tables and then over edge a on the newly reflected shape.

8. Let us apply unfolding to actual billiard paths. Compute the quantities associated with the following degenerate trajectories:



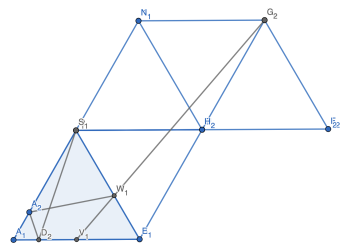
(A): **[5]** The value of $\tan(\theta)$ on the 3-Billiards Table. Note that the trajectory starts at the midpoint of the base. Remember the triangle has unit side lengths.

Solution to Problem 8:



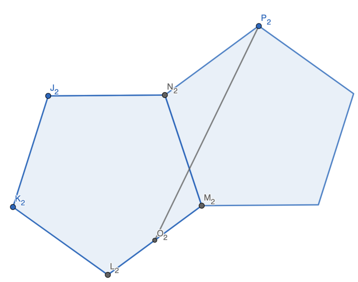
(B): **[5]** The length of the above trajectory on the 5-Billiards table. Note that the trajectory starts at the midpoint of the base. **Note:** $\cos(108°) = \frac{1-\sqrt{5}}{4}$. Remember the pentagon has unit side lengths!

(a) We apply unfolding:



Since $\tan\theta$ is the opposite over adjacent, we simply need to determine those two quantities. We see that the opposite is two triangle heights or $2 \cdot \frac{\sqrt{3}}{2}$. The adjacent is 1.5 lengths of the base which is $\frac{3}{2}$. Thus our tangent is $\boxed{\frac{2\sqrt{3}}{3}}$.

(b) Unfolding we have:



From here it is clear that there are two similar triangles in ratio 1 : 2. Thus the length of the trajectory is three times the length of the segment between the two white dots. We use the law of cosines to compute this value as:

$$s^2 = \frac{1}{4} + \frac{1}{9} - 2\frac{1}{6}\cos(108°)$$

This yields $s = \frac{1}{6}\sqrt{10+3\sqrt{5}}$ so our answer is $3s$ or $\boxed{\dfrac{1}{2}\sqrt{10+3\sqrt{5}}}$

9. **[10]** Consider table T and edge e. Consider the unfolding transformation $U : T \to R$ that reflects T over edge e and then deletes edge e. The resulting reflected table that consists of two copies of T, with one reflected, is R.

   Now consider a periodic path P on table T that has combinatorial period k, and hits edge e exactly n times. Determine, with proof, the combinatorial period of $U(P)$ on R. You may assume T is a convex polygon with pockets solely at its corners.

   Solution to Problem 9:

   First, we partition R into T and T' where T' is the reflected copy of T. Suppose P hits edge e at times $t_1...t_n$. There are two cases, depending on whether n is even or odd. Let the period of P be C.

   Upon reflection, if n is even we see that the unfolded path $U(P)$ must alternate between T and T' an even amount of times precisely at the times where it hits edge e or $t_1...t_n$. In this case, since the path is periodic and follows the law of reflection, upon hitting t_n, $U(P)$ must follow the same path as P from time t_n to C, thus it has the same period as P. In this case, the combinatorial period is $k - n$ since edge e does not exist on R and we have to remove n edges.

   If n is odd, then we still hit all $k - n$ points, but at time C we are on copy T', and thus at the corresponding point of the starting point $P(0)$ on T'. Thus, by symmetry of the construction, continuing on for another $k-n$ hits, we will reach the original $P(0)$ on T. Thus the combinatorial period is $2k - 2n$.

We can repeatedly unfold a shape until it becomes a larger, more familiar billiards table. It so turns out that the equilateral triangle unfolds into a regular hexagon, so extending a trajectory on the equilateral triangle, we have:

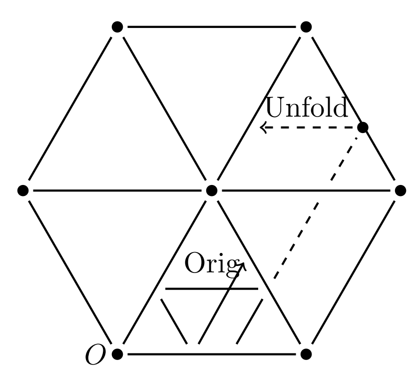


Figure 6: The Unfolded Trajectory of an Equilateral Triangle

10. Demonstrate how to unfold the following billiard tables into their corresponding shapes. Give a diagram that explains how this is done. This need not be rigorous.

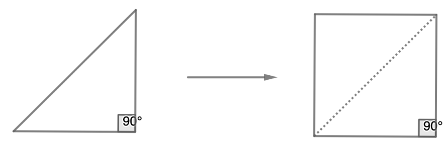
    (a) **[3]** A $45° - 45° - 90°$ triangle to a square.

    (b) **[3]** A $36° - 90° - 54°$ triangle to a pentagon.

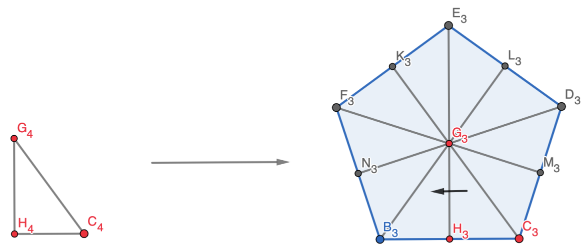
    (c) **[4]** A $36° - 36° - 108°$ triangle to a 5-pointed star with $72°$ internal angles and $144°$ external angles.

Solution to Problem 10:

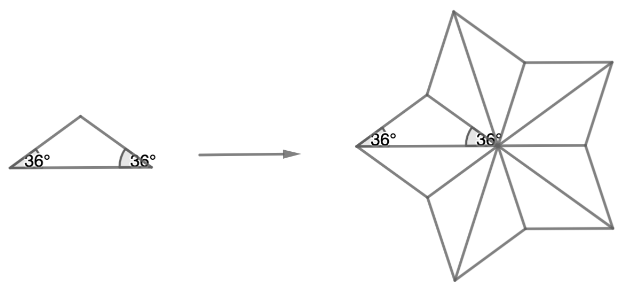
(a) Solution:



(b) Solution:



(c) Solution:



11. Suppose you have a billiards table T that is a $\left(\frac{180}{n}\right)^\circ - 90^\circ - \left(\frac{90n-180}{n}\right)^\circ$ right triangle with pockets at its corners. Orient the triangle such that the side directly opposite the $\left(\frac{180}{n}\right)^\circ$ angle is the base.

    (a) **[3]** Show that T can be unfolded into a regular n-gon.

    (b) **[3]** Show that if n is even, then starting from any point on the base (not including pockets) of T, any non-degenerate trajectory moving vertically upward is periodic.

    (c) **[4]** Determine the combinatorial period of the above trajectory in terms of n where n is even. Justify your answer.

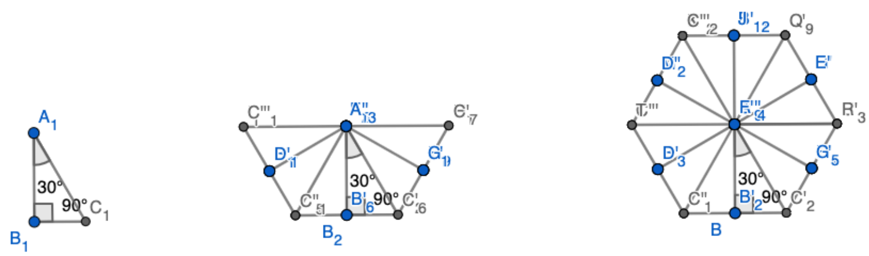
Solution to Problem 11:

(a) **Solution 1:** We start by unfolding the triangle along its longest leg. Then, we take the reflected triangle and unfold it along its hypotenuse. We continue alternating between the longest leg and the hypotenuse until we do a total of n unfoldings. Doing this repeated operation rotates the triangle around the apex point across from the original triangle's longest leg. In fact, after unfolding n times, this angle is now 180°. Thus, we have unfolded to half of a regular n-gon. At this point, we delete all the internal edges we have unfolded.

Next, we unfold over the horizontal edge to give us the full n-gon.

Solution 2:

Another way to do this is to unfold the triangle into a **half**-n-gon and then unfold the half n-gon into the full n-gon. This is cone by unfolding along the longest leg and hypotenuse repeatedly until the central angle reaches 180°. We know this can be done since the angle itself is $\frac{180}{n}$ degrees and so unfolding n times along that angle yields the result. Then, we unfold along the large edge created by the 180 degree unfolding to create the regular n-gon.



(b) Given that we can unfold the triangle into a regular n-gon and n is even, any non-degenerate trajectory will hit the top edge of the regular n gon and come right back down with a period of 2 on the unfolded n-gon. Since periodic trajectories on unfolded tables correspond to periodic trajectories on the table itself, thetrajectory on the triangle is periodic.

(c) First, since we are moving vertically upwards, if this trajecotry is periodic, it must come back onto itself in order to be consistent with the law of reflection. Similarly, in order to come back onto itself, it must reflect off of another horizontal line. We can use unfolding to determine when the **first** horizontal line will be.

Let us iteratively fold the table T as we move up along the base. Doing this starts to produce the regular n-gon folding discussed in parts a and b. As we unfold moving vertically, the path stays straight while the triangle unfolds as we cross an edge. The first time this path hits a horizotnal line is exactly at the half n-gon diameter of the polygon. It then comes immediately back down for the process to be repeated again.

On the unfolded half n-gon the period is 2. It then reflects back down onto the base. If we included the reflected edges, along the way it hits $\frac{n}{2} - 1$ edges going both up and down. Finally we must include the point at which it hits the top edge of the half n-gon and the time at which it returns to the base, for a total combinatorial period of $\boxed{n}$.

Challenge Problems

These problems all use techniques and tools built up in the power round as main ideas, but you will have to use your own creativity to finish them off. Good luck!

12. (a) **[5]** Show that a non-degenerate trajectory on the 4-billiards table is periodic if and only if it has a rational or undefined slope.

(b) **[4]** Suppose the slope of a non-degenerate trajectory on the 4 billiards table is rational and can be written as the reduced fraction $\frac{a}{b}$. Determine, with proof, its combinatorial period (in terms of a and b).

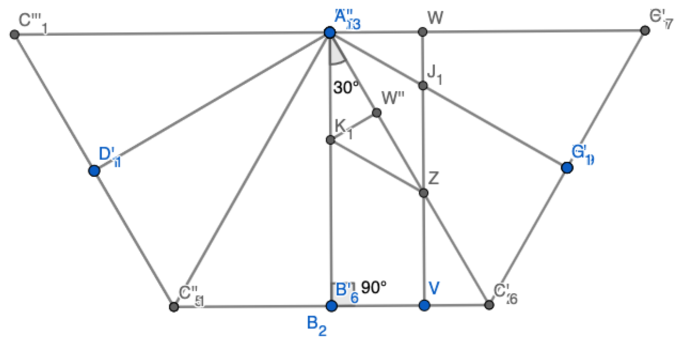


Figure 7: Here is an example of a periodic path of period 6 on a regular hexagon and its corresponding path on the triangle

(c) [**1**] Compute the combinatorial period of a non-degenerate trajectory with slope $\frac{2020}{2021}$.

Solution to Problem 12:

(a) First we prove the if direction. If the slope is undefined, then the path must be vertical and we'll just bounce up and down with period 2. Else the slope is rational. WLOG assume the start point is on an edge (we can do this because we can just follow the trajectory until we hit an edge). Say we start at $(x, 0)$.

Do a BIG number of unfoldings. We can see that any reflected square shifted from the original square by an even number of horizontal and vertical shifts will be in the same orientation as the original square, with the "new starting point" at a location of $(x+2i, 2j)$ for integers i, j. Then, assuming the slope is $\frac{m}{n}$ we pass through $(x+2n, 2m)$, and at this point the path must repeat itself because the orientation of this shifted square is the same as that of the original square. Thus, the trajectory is periodic.

For the converse, do the BIG unfoldings again. Because the trajectory is periodic, with period C, after time C, we must reach the starting point of a reflected square with the same orientation as the original, say $(x + 2n, 2m)$. Then the path we take can be considered to be a straight line from $(x, 0)$ to $(x + 2n, 2m)$ and thus the slope is either undefined (vertical) or rational.

(b) Since we are going from $(x, 0)$ to $(x + 2a, 2b)$ we are hitting exactly $2|a|$ vertical edges and $2|b|$ horizontal edges (taking into account that a or b could be negative), for a total combinatorial period of $\boxed{2|a| + 2|b|}$.

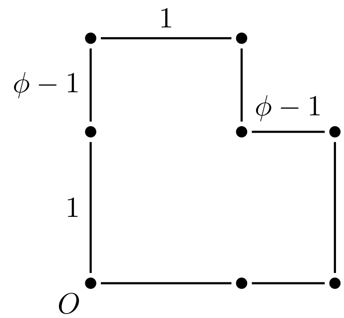
(c) The answer is $2 \cdot 2020 + 2 \cdot 2021 = \boxed{8082}$

13. The **Golden L** is a billiards table that looks as follows:

[**10**] Show that any periodic trajectory on the golden L has either an undefined slope, or a slope of the form $a + b\phi$ where $a, b \in \mathbb{Q}$ where $\mathbb{Q}$ is the rational numbers and ϕ is the golden ratio ($\frac{1+\sqrt{5}}{2}$). As before pockets are black dots. **Hint:** Rewind and track the horizontal and vertical distances separately.

Solution to Problem 13:

Similar to the last problem, we see that if the path is horizontal or vertical then we are done with a period of 2. So, WLOG assume that the slope is non-zero and not undefined. As a



result, it has a defined reciprocal.

Now, let us imagine unfolding the Golden L as we travel along a path. This yields a straight ray that represents the path. Eventually, as we unfold we will reach a point that corresponds to the original point, but after a single period. Since all the sides are either horizontal or vertical, the horizontal distance traveled within the shape is the same as the horizontal distance traveled on the ray, and similar with the vertical distance.

The slope of this line is therefore the absolute value of the vertical distance traveled over the absolute value of the horizontal distance traveled. We only need compute these two quantities to calculate the slope.

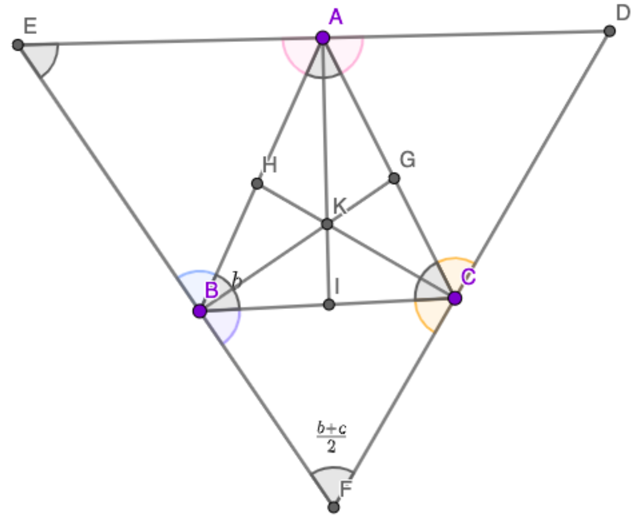
Since the slope is non-vertical, it must hit the left-most edge. Therefore, we rewind the path until we hit the edge. From the left most edge, we can hit either of the two rightmost edges so we travel either 1 or ϕ distance. Similarly once we hit the rightmost edges we travel either 1 or ϕ distance to get to the left edge. So therefore the vertical distance traveled is $\alpha_1 + \beta_1\phi$ for some integer constants α_1, β_1. By symmetry the vertical distance traveled is also $\alpha_2 + \beta_2\phi$ for some integer constants α_2 and β_2.

Therefore the slope is $\frac{\alpha_1+\beta_1\phi}{\alpha_2+\beta_2\phi}$. Multiplying by $k = \phi - \frac{\beta_2+\alpha_2}{\beta_2}$ by the expression on denominator, we get $\beta_2\phi^2 - \beta_2\phi - \alpha_2\phi + \alpha_2\phi + c$ for some rational constant c. This is $\beta(\phi^2 - \phi) + c$ which is $\beta + c$, rational number, since $\phi^2 - \phi = 1$.

Thus, if we multiply by k on both the top and bottom we get a number of the form $a + b\phi$ where a and b are rational numbers and we are done.

14. **[10]** Let T be a billiards table that is an acute triangle with pockets at its corners. Show that T contains exactly two periodic trajectories of combinatorial period 3. Note that trajectories that are rewinds or fast-forwards of each other are considered the same in this case.

Solution to Problem 14:



First, if T is a periodic trajectory with combinatorial period 3., it must be an inscribed triangle with vertices on separate edges of the triangle. Otherwise, T would hit the same edge twice in a row, which is not possible, since then it would hit a pocket and be degenerate.

Thus we may assume T corresponds to ABC a triangle. Thus, WLOG, assume A on $\overline{DE}$, B on $\overline{EF}$ and C on $\overline{FD}$ with angles a, b, c being the measures of $\angle BAC, \angle ACB, and \angle BCA$. We will show uniqueness of this triangle in $\triangle DEF$. The triangle shape will then give exactly two solutions corresponding to the clockwise and counterclockwise methods of traversal.

We will first show that $\triangle ABC$ is the orthic triangle of $\triangle DEF$ (i.e the triangle made up of the foot of the altitudes). Since the orthic triangle is unique via construction for any given triangle, this will show that the periodic path is unique.

To show that $\triangle ABC$ is orthic , we simply need to show that $\overline{DB}$ is an altitude of $\triangle DEF$. Then, by symmetry the same will hold for the altitudes. First, since $\triangle ABC$ is a periodic trajectory, it follows the law of reflection. Thus, the angle bisector of A is perpendicular to $\overline{DE}$ and similarly for the other sides.

Let the incenter of $\triangle ABC$ be K which is the concurrent intersection point of angle bisectors of $\triangle ABC$. Consider quadrilateral $DAKC$. It has right angles at A and C, so this is a cyclic quadrilateral.

Thus, $\angle KAC = \angle DKC = \frac{a}{2}$. Thus angle $\angle DKC$ is $90 - \frac{a}{2}$. We now compute $\angle CKB$. This by angle chasing is simply $180 - \frac{b+c}{2} = 180 - \frac{180-a}{2} = 90 + \frac{a}{2}$ Thus, $\angle DKB$ is 180 and so since $\overline{KB}$ is perpendicular to $\overline{EF}$ so is $\overline{DB}$. Thus, $\overline{DB}$ is an altitude and the triangle is the orthic triangle.

Now, we have shown that the triangle traced out by the periodic trajectory is equivalent to the orthic triangle of the acute $\triangle DEF$. We simply need to show such an orthic triangle exists. However, this is easy. Since $\triangle DEF$ is acute, the altitudes of $\triangle DEF$ are fully contained in $\triangle DEF$. Thus, by convexity of a triangle, the orthic triangle exists.

Thus, the periodic path exists and is unique.

15. For this problem, we will need a couple of tools:

Definition. Define a **simple billiards table** be a convex polygon with pockets at its corners.

Theorem. *For any simple billiards table, if you are given a line segment that starts at point P_1 on e_1 and ends at point P_2 on e_2, and P_1, P_2 are not pockets, then there exists an ϵ such that if you translate $\overline{P_1P_2}$ in a direction perpendicular to the slope of $\overline{P_1P_2}$ by any $t < \epsilon$ and extend/crop it to line segment $\overline{P_1^tP_2^t}$ with P_1^t on e_1 and P_2^t on e_2, the line $\overline{P_1^tP_2^t}$ never contains a pocket.*

[10] Show that for any periodic trajectory with odd combinatorial period p on a simple billiards table, there exists a periodic trajectory with combinatorial period $2p$.

Solution to Problem 15:

Solution 1: Let the periodic trajectory be $T_0 \dots T_p$ where $T_p = T_0$. Let T_0 be on line segment $\overline{AB}$ and orient the table so $\overline{AB}$ lays horizontally. Since the trajectory is periodic, it cannot lay entirely on an edge (since otherwise it would hit a pocket and be degenerate). Thus the trajectory points at a different direction than horizontal.

Imagine traveling along the periodic path, unfolding the table as you go until you return back to the corresponding starting point. Let the corresponding points to $T_0 \dots T_p$ on the unfolded trajectory be $Q_0 \dots Q_p$. Since the combinatorial period is odd, we would have unfolded an odd number of times. Entering the shape after an odd number of unfoldings, which are the same as reflections, reveals that the shape has a reverse orientation. That is, the vertices of the billiards table go clockwise instead of counter clockwise. Since Q_p corresponds to Q_0, we let Q_p be on line segment $\overline{A'B'}$.

Since we had unfolded along the path, we see that the path itself did not change direction, so we see that the reverse orientation is exactly the orientation that we would get if we reflected the original billiards table over the line T_0T_1.

Now, let us continue unfolding for another p times yielding $Q_{p+1} \dots Q_{2p}$ until we return to the original point a second time. This time, however since we have reflected $2p$ times, the orientation is the same as the original billiards table. So let Q_{2p} be on $\overline{A''B''}$.

Now, each $\overline{Q_iQ_{i+1}}$ is a segment on the line $\overline{Q_0Q_{2p}}$ and is also a singular segment on the ith reflected shape. As such by the theorem stated in the problem, if we shift each segment separately there exists $\epsilon_0 ... \epsilon_{2p-1}$ such that for $t < \epsilon_i$, $\overline{Q_iQ_{i+1}}$ can be translated perpendicularly without containing a pocket.

Let ϵ^* be the minimum of all the ϵ_i. Then, suppose we translate the extended line $\overline{Q_0Q_{2p}}$ orthogonally to the right by some $t < \epsilon^*$, to $\overline{R_0R_{2p}}$ towards point A. This line hits the same edge of the same shape at points $\overline{R_0R_{2p}}$, such that $R_0 = R_{2p}$. Thus, this trajectory has a combinatorial period of at most $2p$. We need now show that it does not have a period of p.

Consider R_p which was orthogonally transported from Q_p. The orientation of the table from Q_p to Q_{p+1} is flipped and so is the orientation of $\overline{A'B'}$ relative to $\overline{AB}$. In fact, $\overline{A'B'}$ is $\overline{AB}$ flipped over $\overline{Q_0Q_p}$. Thus translating $\overline{Q_0Q_p}$ to the shits the line towards B but since $\overline{A'B'}$ is reflected, it shifts R_p towards A'. Thus there is no correspondence between R_p and R_0 since they are on different positions on the table. So the period can no longer be p.

Therefore there exists a path with combinatorial period $2p$.

Solution 2: (From CCA #1)

Suppose the reflections that the ball undergoes in one period are at point Q_1 on side l_1,Q_2 on side $l_2, \dots$ and Q_p at l_p, in that order. For all $k \in \{1, 2, ..., p\}$, let θ_k be the angle between the path of the ball (taken either immediately before or after reflecting off of l_k, which are the same by the law of reflection). For the remainder of this proof, consider the indices of l, Q , and θ in modulo p.

Suppose the polygon is $P_1P_2 \dots P_n$, where vertices are listed in counterclockwise order. Direct the lengths of segments on the perimeter such that if $\overline{XY}$ is a subsegment of P_iP_{i+1} , where indices are taken modulo n,$XY = \epsilon|XY|$, where $\epsilon = 1$ if XY and P_iP_{i+1} point in the same direction, and $\epsilon = -1$ otherwise. For the remainder of the proof, consider the indices of P modulo n. Now, fix a small constant d that we will specify later, take the sequence of points R_1, $R_2 \dots R_{2p}$ such that R_k is on l_k and $\overline{Q_kR_k} = (-1)^k \frac{d}{\sin(\theta_k)}$.

Note that all of these points are distinct, since if R_i and R_j with $i < j$ are on the same side, then we have $j = i + p$, implying that $\overline{Q_iR_i}$ and $\overline{Q_jR_j} = \overline{Q_iR_j}$ have different signs. Consider the indices of R in modulo $2p$ for the remainder of the proof.

We claim that $\overline{R_aR_{a+1}}||\overline{Q_aQ_{a+1}}$ for all a. Since a and $a+1$ have different parities, Q_aR_a and $Q_{a+1}R_{a+1}$ have different signs, so R_a and R_{a+1} are on opposite sides of Q_aQ_{a+1} . Furthermore, the distances from R_a and R_{a+1} to Q_aQ_{a+1} are $|Q_aR_a|\sin(\theta_a) = d$ and $|Q_{a+1}R_{a+1}|\sin(\theta_{a+1}) = d$, implying that they are parallel. Also, note that this doesn't depend on a, so if we take the value of ϵ given by the given theorem for the union of the segments in the path with reflections at the Q_i, it suffices to take $d < \epsilon$.

Now, since l_a externally bisects $\angle Q_{a-1}Q_aQ_{a+1}$ and we have that $Q_{a-1}Q_a||R_{a-1}R_a$ and $Q_{a+1}Q_a||R_{a+1}R_a$, we get that l_a externally bisects $\angle R_{a-1}R_aR_{a+1}$ for all a. This implies that the path starting at some point on R_1R_2 and starting to move along R_1R_2 only has reflections at the points $R_1, R_2, \dots, R_{2p}$ once each, implying the result.