1. Julia and James pick a random integer between 1 and 10, inclusive. The probability they pick the same number can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: 11

   Solution: Regardless of what number Julia picks, James has a $\frac{1}{10}$ chance he chooses the same number (that is, the two are independent events). Hence, the probability is $\frac{1}{10}$ and our answer is $\boxed{11}$.

2. There are 38 people in the California Baseball League (CBL). The CBL cannot start playing games until people are split into teams of exactly 9 people (with each person in exactly one team). Moreover, there must be an even number of teams. What is the fewest number of people who must join the CBL such that the CBL can start playing games? The CBL may not revoke membership of the 38 people already in the CBL.

   Answer: 16

   Solution: Since $38/9 > 4$, there must be at least 6 teams for games to start in the CBL. Thus, the minimum of people that need to join is $6 \cdot 9 - 38 = \boxed{16}$.

3. An ant is at one corner of a unit cube. If the ant must travel on the box's surface, the shortest distance the ant must crawl to reach the opposite corner of the cube can be written in the form $\sqrt{a}$, where a is a positive integer. Compute a.

   Answer: 5

   Solution: If we unfold the cube we see that ant must travel diagonally across a 1×2 rectangle, thus the shortest length is $\sqrt{5}$, and $a = \boxed{5}$.

4. Let $p(x) = 3x^2 + 1$. Compute the largest prime divisor of $p(100) - p(3)$.

   Answer: 103

   Solution: We have $p(100) - p(3) = 3(100^2 - 3^2) = 3 \cdot 97 \cdot 103$, so the answer is $\boxed{103}$.

5. Call a positive integer *prime-simple* if it can be expressed as the sum of the squares of two distinct prime numbers. How many positive integers less than or equal to 100 are prime-simple?

   Answer: 6

   Solution: The integers $2^2 + 3^2$ through $2^2 + 7^2$, $3^2 + 5^2$, $3^2 + 7^2$, and $5^2 + 7^2$, or 13, 29, 58, 34, 58, and 74, are all prime-simple, which yields a total of $\boxed{6}$ prime-simple integers under 100.

6. Jack writes whole numbers starting from 1 and skips all numbers that contain either a 2 or 9. What is the 100^{th} number that Jack writes down?

   Answer: 155

   Solution: Skipping all numbers with 2 or 9 is the same as counting in base 8. We convert 100 into base 8 to obtain 144_8. Then because the numbers of the form $12\_$ are skipped, as well as 152_{10}, the one hundredth number he writes down is $144 + 11 = \boxed{155}$.

7. A fair six-sided die is rolled five times. The probability that the five die rolls form an increasing sequence where each value is strictly larger than the one that preceded can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: 1297

Solution: There are $\binom{6}{5} = 6$ distinct increasing sequences of length five. There are 6^5 different possible sequences of five die rolls. The probability is therefore $\frac{6}{6^5} = \frac{1}{6^4} = \frac{1}{1296}$, and our answer is $\boxed{1297}$.

8. Let $ABCD$ be a unit square and let E and F be points inside $ABCD$ such that the line containing $\overline{EF}$ is parallel to $\overline{AB}$. Point E is closer to $\overline{AD}$ than point F is to $\overline{AD}$. The line containing $\overline{EF}$ also bisects the square into two rectangles of equal area. Suppose $[AEFB] = [DEFC] = 2[AED] = 2[BFC]$. The length of segment $\overline{EF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: 4

Solution: Since EF is parallel to AB, $AEFB$ is a trapezoid. The area of the trapezoid $AEFB$ is $\frac{1}{3}$ since $\text{area}(AEFB) + \text{area}(DEFC) + \text{area}(AED) + \text{area}(BFC) = 3\text{area}(AEFB) = \text{area}(ABCD) = 1$, and since EF bisects the square into two rectangles of equal area, the height of the trapezoid $AEFB$ is $\frac{1}{2}$. Thus, $\frac{1}{4}(1 + EF) = \frac{1}{3}$, so $EF = \frac{1}{3}$, and our answer is $\boxed{4}$.

9. A sequence a_n is defined by $a_0 = 0$, and for all $n \geq 1$, $a_n = a_{n-1} + (-1)^n \cdot n^2$. Compute a_{100}.

Answer: 5050

Solution: We have $a_{n+2} = a_{n+1} + (-1)^{n+2} \cdot (n+2)^2 = a_n + (-1)^{n+1} \cdot (n+1)^2 + (-1)^{n+2} \cdot (n+2)^2$. If n is even, then $a_{n+2} = a_n - (n+1)^2 + (n+2)^2 = a_n + 2n + 3$. Thus, $a_{100} = 3 + 7 + 11 + \cdots + 199 = 101 \cdot 50 = \boxed{5050}$.

10. How many integers $100 \leq x \leq 999$ have the property that, among the six digits in $\lfloor 280 + \frac{x}{100} \rfloor$ and x, exactly two are identical?

Answer: 294

Solution: We know that the units digit of $\lfloor 280 + \frac{x}{100} \rfloor$ is the hundreds digit of x, so all other digits must be distinct (and not equal to the hundreds digit of x). Note that $\lfloor 280 + \frac{x}{100} \rfloor$ will always begin with 28 for $x < 1000$. Let $x = 100p + 10q + r$ for some integers p, q, and r, with $0 \leq p, q, r \leq 9$. Then we have $\lfloor 280 + \frac{x}{100} \rfloor = \overline{28p}$ and $x = \overline{pqr}$. Then $p, q, r \neq 2, 8$, and p, q, r must also be pairwise distinct. This leaves us with 7 choices for p, 7 choices for q, and 6 choices for r, for $\boxed{294}$ choices for x in total.

11. Compute $\displaystyle\sum_{x=1}^{999} \gcd(x, 10x + 9)$.

Answer: 2331

Solution: When x is a multiple of 3 but not 9, $\gcd(x, 10x + 9) = \gcd(x, 9) = 3$ by Euclidean Algorithm. Similarly, when x is a multiple of 9, $\gcd(x, 10x + 9) = 9$. In all other cases, $\gcd(x, 10x + 9) = 1$. There are 666 values of x in the third case, 222 in the first case, and 111 in the second case, giving a sum of $1 \cdot 666 + 3 \cdot 222 + 9 \cdot 111 = \boxed{2331}$.

12. A hollow box (with negligible thickness) shaped like a rectangular prism has a volume of 108 cubic units. The top of the box is removed, exposing the faces on the inside of the box. What is the minimum possible value for the sum of the areas of the faces on the outside and inside of the box?

Answer: 216

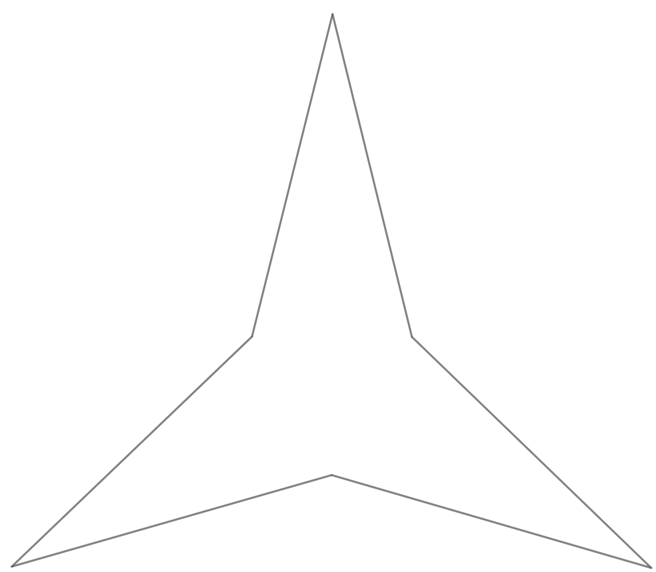
Solution: If the side lengths of the box are a, b, and c, then the new surface area is (WLOG) $4ab + 4bc + 2ac$. By AM-GM, $\frac{4ab+4bc+2ac}{3} \geq (32(abc)^2)^{1/3} = 72$. Thus, the answer is $72 \cdot 3 = \boxed{216}$.

13. Compute the expected sum of elements in a subset of $\{1, 2, 3, \ldots, 2020\}$ (including the empty set) chosen uniformly at random.

Answer: 1020605

Solution: Each element is included in 2^{2019} of the subsets, and we must divide by 2^{2020} total subsets, making the expected value of the sum equal to $\frac{1+2+3+\ldots+2020}{2} = \frac{2020 \cdot 2021}{4} = 505 \cdot 2021 = \boxed{1020605}$.

14. In the star shaped figure below, if all side lengths are equal to 3 and the three largest angles of the figure are 210 degrees, its area can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and that b is square-free. Compute $a + b + c$.



Answer: 14

Solution: We split the figure into one small equilateral triangle and three $30-75-75$ triangles. The area of the $30-75-75$ triangles is $\frac{1}{2}3 \cdot 3\sin(30°) = \frac{9}{4}$. The side length of the equilateral triangle is $2 \cdot 3\sin(15°) = \frac{3(\sqrt{6}-\sqrt{2})}{2}$. Thus, the total area is

$$3 \cdot \frac{9}{4} + \frac{\left(\frac{3(\sqrt{6}-\sqrt{2})}{2}\right)^2 \sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$$

and our answer is $\boxed{14}$.

15. Consider a random string s of 10^{2020} base-ten digits (there can be leading zeroes). We say a substring s' (which has no leading zeroes) is *self-locating* if s' appears in s at index s' where the string is indexed at 1. For example the substring 11 in the string "122352242411" is self-locating since the 11th digit is 1 and the 12th digit is 1. Let the expected number of self-locating substrings in s be G. Compute $\lfloor G \rfloor$.

Answer: 1817 OR 2019

Solution:

We solve this problem using linearity of expectations and random variables. Let X_i be an indicator of the event that there exists a self-locating substring at index i. It will take the value 1 if there is a self-locating substring and 0 if not. Then, the number of substrings can be thought of as $N = \sum_{i=1}^{10^{2020}} X_i$, where N is a random variable representing the number of self-locating substrings. By linearity of expectation, $E(N) = \sum_{i=1}^{10^{2020}} E(X_i)$. We see that the expected value of X_i is simply the probability that there is a self-locating substring at i.

We now compute the probability that at index i, there exists a self-locating substring. Note, if the index is between 10^k and 10^{k+1} then string must have exactly $k+1$ digits. So the probability that the next $k+1$ digits match up with the index is $\frac{1}{10}^{k+1}$ since the values in the string are independent of each other.

Note this does NOT apply to the last 2019 digits of the string (since there won't be enough digits to constitute a 2020-digit number). However, let us ignore that for now and compute as if they would.

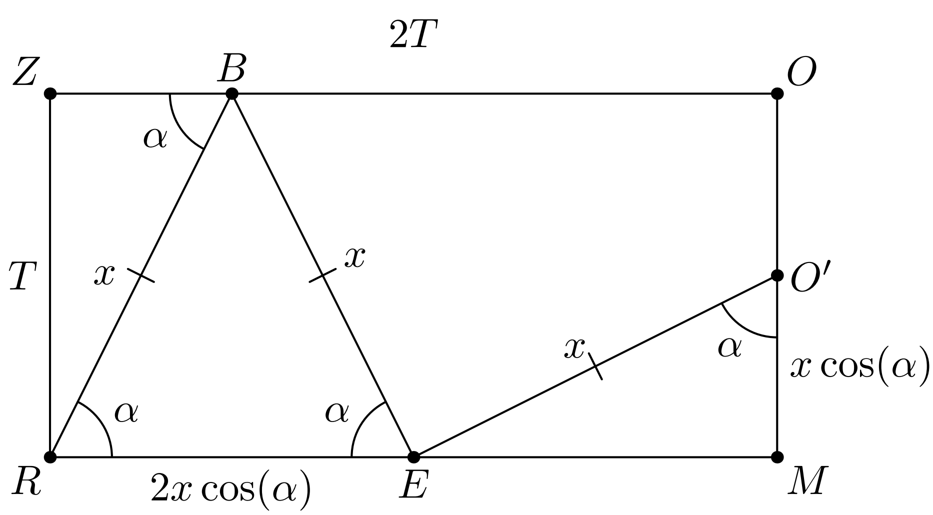
We see that over all $9 \cdot 10^k$ numbers between 10^k and 10^{k+1} the sum of the expected values is $\frac{9}{10}$. So we sum $\frac{9}{10}$, 2020 times to obtain 1818. However, since we know that the expected value will be slightly less than 1818, the floor of our actual answer is $\boxed{1817}$.

Note: During the tournament, we erroneously gave a clarification that leading zeros were allowed in the string s'. In this case, all strings could be prepended with zeros, thus giving an extra 10/9 multiplier to all strings. This gives a final answer of 2019. We thus gave credit to both 1817 and 2019.

16. Let T be the answer to question 18. Rectangle $ZOMR$ has $ZO = 2T$ and $ZR = T$. Point B lies on segment ZO, O' lies on segment OM, and E lies on segment RM such that $BR = BE = EO'$, and $\angle BEO' = 90°$. Compute $2(ZO + O'M + ER)$.

Answer: 21

Solution:



We first solve the problem in terms of T by computing some lengths. Let $RB = BE = EO' = x$ and $\angle BRE = \alpha$. By angle chasing, we have that $\angle BRE = \angle BER = \angle EO'M$, so we have that $O'M = x\cos\alpha$ and $ER = 2x\cos\alpha$. Thus, the quantity we which to compute is $4T + 6x\cos\alpha$.

We now relate x in terms of T. Again by angle chasing, we have that $\angle ZBR = \angle BRE = \alpha$, so $T = ZR = x\sin\alpha$. Moreover, we have that $2T = RM = RE + EM = 2x\cos\alpha + x\sin\alpha$. Subtracting the two equations, we have that $T = 2x\cos\alpha$, so $3x\cos\alpha = \frac{3}{2}T$. Thus, the quantity we want is $4T + 3T = 7T$.

Looking at the possible answers from question 18, we know that $T = \frac{15}{2}, 6$, or 3, which correspond to the answer to this problem being $\frac{105}{2}, 42$, or 21. Looking at the other two questions, the only answer that gives consistent answers to all three questions is $\boxed{21}$.

17. Let T be the answer to question 16. Compute the number of distinct real roots of the polynomial $x^4 + 6x^3 + \frac{T}{2}x^2 + 6x + 1$.

Answer: 4

Solution: Clearly, $x = 0$ is not a solution. Thus, we can manipulate the polynomial as follows:

$$\begin{aligned} x^4 + 6x^3 + \frac{T}{2}x^2 + 6x + 1 &= 0 \\ x^2 + 6x + \frac{T}{2} + 6\frac{1}{x} + \frac{1}{x^2} &= 0 \\ \left(x + \frac{1}{x}\right)^2 - 2 + 6\left(x + \frac{1}{x}\right) + \frac{T}{2} &= 0 \end{aligned}$$

Let $y = x + \frac{1}{x}$. Then we wish to solve the equation $y^2 + 6y + (\frac{T}{2} - 2) = 0$.

Since we only care about real solutions, we look for values of y such that $x + \frac{1}{x} = y$ has real solutions. Since $x = 0$ is not a solution, we multiply by x to get the quadratic equation $x^2 - yx + 1 = 0$. For this to have real solutions of x, the discriminant must be nonnegative, so we must have that $y^2 - 4 \geq 0$, or $|y| \geq 2$.

By the quadratic equation, the solutions to $y^2 + 6y + (\frac{T}{2} - 2) = 0$ are $y = \frac{-6\pm\sqrt{6^2-4(\frac{T}{2}-2)}}{2} = -3 \pm \sqrt{11 - \frac{T}{2}}$. Since $-3 - \sqrt{11 - \frac{T}{2}}$ is always less than -2 as long as $T \leq 22$, so if $T \leq 22$, this solution for y contributes 2 real solutions for x, and if $T > 22$, then this solution for y contributes no real solutions for x.

We now apply casework on $y = -3 + \sqrt{11 - \frac{T}{2}}$. Considering y as a function of T, notice that we have the following relationship:

$$\begin{cases} y \text{ not real} & T > 22 \\ y < -2 & 20 < T \leq 22 \\ y = -2 & T = 20 \\ -2 < y < 2 & -28 < T < 20 \\ y = 2 & T = -28 \\ y > 2 & T < -28 \end{cases}$$

When $y = -2$ or 2, the discriminant of the quadratic in x is 0, so there is 1 real solution for x from this value of y. When $-2 < y < 2$ and when y is not real, there are no real solutions from this value of y. Finally, the other cases give two real solutions from the value of y. Combining our results with the work above, we get the following relationship between the number of real solutions and T:

$$\begin{cases} 0 & T > 22 \\ 2 & T = 22 \\ 4 & 20 < T < 22 \\ 3 & T = 20 \\ 4 & -28 < T < 20 \\ 3 & T = -28 \\ 4 & T < -28 \end{cases}$$

A special case to take care of here is $T = 22$; the two solutions for y are actually the same, so we must be careful not to double count the roots.

We now send all possible values of the answer to question 18 to see which values work; it turns out that $T = 21$, so the answer to this question is $\boxed{4}$.

18. Let T be the answer to question 17, and let $N = \frac{24}{T}$. Leanne flips a fair coin N times. Let X be the number of times that within a series of three consecutive flips, there were exactly two heads or two tails. What is the expected value of X?

Answer: 3

Solution: We first solve the problem in terms of N.

Leanne flips the fair coin N times, so there are $N-2$ sets of three consecutive flips. For any set of three consecutive flips, the probability that there are exactly two heads or two tails is $\frac{2 \cdot 3}{2^3} = \frac{3}{4}$. Then by linearity of expectation, we have that $E[X] = \frac{3(N-2)}{4}$.

Looking at the possible answers to question 17, we know that T is 0, 2, 3, or 4. Clearly, $T \neq 0$, so we test $T = 2$, $T = 3$, and $T = 4$. The associated values of N are $12, 8$, and 6, respectively, so the associated values of $E[X]$ are $\frac{15}{2}, 6$, and 3 respectively. We now send these answers to question 16 to see which one gives consistent answers to all three problems, and it turns out that the correct answer for T is 4. Thus, the answer to this problem is $\boxed{3}$.

19. John is flipping his favorite bottle, which currently contains 10 ounces of water. However, his bottle is broken from excessive flipping, so after he performs a flip, one ounce of water leaks out of his bottle. When his bottle contains k ounces of water, he has a $\frac{1}{k+1}$ probability of landing it on its bottom. What is the expected number of number of flips it takes for John's bottle to land on its bottom?

Answer: 6

Solution: By linearity of expectation, we sum all of $1 \cdot \frac{1}{11}$, $2 \cdot \frac{10}{11} \cdot \frac{1}{10}$, $3 \cdot \frac{10}{11} \cdot \frac{9}{10} \cdot \frac{1}{9}$, up to $11 \cdot \frac{10}{11} \cdot \frac{9}{10} \cdots \frac{1}{1}$. Thus, the answer is $\sum_{i=1}^{11} \frac{i}{12-i} \prod_{j=1}^{i-1} \frac{11-j}{12-j} = \frac{1}{11} + \frac{2}{11} + \cdots + \frac{11}{11} = \boxed{6}$.

20. Non-degenerate quadrilateral $ABCD$ with $AB = AD$ and $BC = CD$ has integer side lengths, and $\angle ABC = \angle BCD = \angle CDA$. If $AB = 3$ and $B \neq D$, how many possible lengths are there for BC?

Answer: 7

Solution: We draw side length AC. Notice now that in triangle $\triangle ABC$, $2\angle BCA = \angle ABC$. Let $\angle BCA = \alpha$.

We extend AB through B to a point E such that $BE = BC$. Thus, we must have that $\angle BEC = \angle BCE$. Moreover, notice that $2\alpha = \angle ABC = \angle BEC + \angle BCE$, so $\angle BEC = \angle BCE = \alpha$, and $\angle ACE = \angle ACB + \angle CBE = 2\alpha$.
Thus, notice that triangles $\triangle ABC$ and $\triangle ACE$ are similar. Then

$$\frac{AB}{AC} = \frac{AC}{AE} \implies AC^2 = (AB)(AE) = AB(AB + BE) = 3(3 + BC).$$

We must have triangle inequality hold as well, so $AC < 3+BC$, $3 < AC+BC$, and $BC < 3+CA$. Since $AC^2 > 3^2$, the second inequality holds, so we only need to check the first and third inequalities.
First, we must have that $AC < 3 + BC = AC^2/3$. This gives $AC > 3$, so this information is redundant.
We also must have that $BC < 3 + AC$, so

$$\begin{aligned} \frac{AC^2}{3} - 3 < 3 + AC &\implies AC^2 - 9 < 9 + 3AC \\ &\implies AC^2 - 3AC - 18 < 0 \\ &\implies (AC - 6)(AC + 3) < 0, \end{aligned}$$

so we must have that $-3 < AC < 6$. Combining all the inequalities, we determine that $3 < AC < 6$.
Converting back into a condition for BC, we have that

$$\begin{aligned} 3 < AC < 6 &\implies 9 < AC^2 < 36 \\ &\implies 3 < AC^2/3 < 12 \\ &\implies 0 < AC^2/3 - 3 = BC < 9. \end{aligned}$$

Since BC is an integer, we have that BC can be any length from 1 to 8. However, when $BC = 6$, our quadrilateral is degenerate. Thus, there are $\boxed{7}$ possible choices for BC.

21. Let $\triangle ABC$ be a right triangle with legs $AB = 6$ and $AC = 8$. Let I be the incenter of $\triangle ABC$ and X be the other intersection of AI with the circumcircle of $\triangle ABC$. Find $\overline{AI} \cdot \overline{IX}$.

Answer: 20

Solution: Let D and E be the tangency points of the incircle with AB and BC, respectively. Let Y be the intersection of lines CI and DE. Note that $\angle CYB$ is a right angle since side BC is a diameter, so Y lies on the circumcircle of ABC and by Power of a Point, $\overline{AI} \cdot \overline{IX} = \overline{CI} \cdot \overline{IY} = \boxed{20}$.
Solution 2:
The radius is $r = \frac{A}{s} = \frac{24}{12} = 2$ so $AI = 2\sqrt{2}$. By the Incenter-Excenter lemma, $\overline{IX} = \overline{BX} = \overline{CX} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$ so $\overline{AI} \cdot \overline{IX} = \boxed{20}$.

22. Suppose that x, y, and z are positive real numbers satisfying

$$\begin{cases} x^2 + xy + y^2 = 64 \\ y^2 + yz + z^2 = 49 \\ z^2 + zx + x^2 = 57 \end{cases}$$

Then $\sqrt[3]{xyz}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: 69

Solution: Let $\triangle ABC$ be a triangle with $AB = 8$, $BC = 7$, and $CA = \sqrt{57}$. Let P be a point in $\triangle ABC$ such that $PA = x$, $PB = y$, and $PC = z$. Notice that the system of equations is equivalent to the geometric configuration described above, with $\angle APB = \angle BPC = \angle CPA = 120°$.

Now notice that by the Law of Cosines, $\angle ABC = 60°$. Then $\angle PAB = 60° - \angle PBA = \angle PBC$, so by AA similarity triangles $\triangle APB$ and $\triangle BPC$ are similar. Therefore $\dfrac{AP}{PB} = \dfrac{BP}{PC}$, or $y^2 = xz$. Thus x, y, and z form a geometric sequence in that order.

Let $r = \frac{y}{x}$. We rewrite the first two equations to get

$$x^2(1 + r + r^2) = 64$$
$$y^2(1 + r + r^2) = 49$$

Then $\dfrac{y}{x} = r = \dfrac{7}{8}$. Finally, we plug in $r = \dfrac{7}{8}$ to get

$$y = \sqrt{\frac{49}{1 + r + r^2}} = \sqrt{\frac{49 \cdot 64}{8^2 + 8 \cdot 7 + 7^2}} = \frac{56}{13},$$

and our answer is $\boxed{69}$.

23. Let $0 < \theta < 2\pi$ be a real number for which $\cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cdots + \cos(2020\theta) = 0$ and $\theta = \frac{\pi}{n}$ for some positive integer n. Compute the sum of the possible values of $n \leq 2020$.

Answer: 1926

Solution: Using the identity $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$, we substitute $z = e^{i\theta}$ and take the real part of both sides to obtain

$$\begin{aligned}
1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) &= \operatorname{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) \\
&= \operatorname{Re}\left(\frac{\left(1 - e^{i(n+1)\theta}\right)\left(1 - e^{-i\theta}\right)}{\left(1 - e^{i\theta}\right)\left(1 - e^{-i\theta}\right)}\right) \\
&= \operatorname{Re}\left(\frac{1 - e^{i(n+1)\theta} - e^{i\theta} + e^{in\theta}}{2 - 2\cos(\theta)}\right) \\
&= \frac{1 - \cos\left((n+1)\theta\right) - \cos(\theta) + \cos(n\theta)}{2 - 2\cos(\theta)} \\
&= \frac{2\sin\left(\frac{2n+1}{2}\theta\right)\sin\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)} + \frac{1}{2} \\
&= \frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} + \frac{1}{2},
\end{aligned}$$

where we use the product-to-sum and the double-angle formulas to simplify the expression. Since $\sin\left(\frac{\theta}{2}\right) \neq 0$, it follows that $\cos(\theta)+\cos(2\theta)+\cdots+\cos(2020\theta) = 0$ precisely when $\frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{1}{2}$, that is, when $\sin\left(\frac{2n+1}{2}\theta\right) = \sin\left(\frac{\theta}{2}\right)$. Plugging in $n = 2020$, we get $\sin\left(\frac{4041}{2}\theta\right) = \sin\left(\frac{1}{2}\theta\right)$. We have that $\sin\left(\frac{4041}{2}\theta\right) = \sin\left(\frac{1}{2}\theta\right)$ is satisfied when $\frac{4041}{2}\theta - \frac{1}{2}\theta = 2020\theta$ is a even multiple of

π, or when $\frac{4041}{2}\theta + \frac{1}{2}\theta = 2021\theta$ is an odd multiple of π.
In the first case, $\frac{2020}{n}$ must be an even integer, so we have that $n \mid 1010$. In the second case, we need that $\frac{2021}{n}$ is an odd integer, so we need that $n \mid 2021$. By the sum of factors theorem, the sum of the factors of $1010 = 2 \cdot 5 \cdot 101$ is $(2^1 + 2^0)(5^1 + 5^0)(101^1 + 101^0) = 3 \cdot 6 \cdot 102 = 1836$. We also include factors of 2021 that are not factors of 2020 and are less than or equal to 2020; this gives $n = 43$ and $n = 47$. This makes our final sum $1836 + 43 + 47 = \boxed{1926}$.

24. For positive integers N and m, where $m \le N$, define

$$a_{m,N} = \frac{1}{\binom{N+1}{m}} \sum_{i=m-1}^{N-1} \frac{\binom{i}{m-1}}{N-i}.$$

Compute the smallest positive integer N such that

$$\sum_{m=1}^{N} a_{m,N} > \frac{2020N}{N+1}.$$

Answer: 8078

Solution: I claim that

$$a_{m,N} = \frac{1}{\binom{N+1}{m}} \sum_{i=m-1}^{N-1} \frac{\binom{i}{m-1}}{N-i} = \frac{m}{N+1} \sum_{i=m}^{N} \frac{1}{i}.$$

Consider the situation where there are $N+1$ possible candidates for secretary, and the candidates are ranked from 1 to $N+1$, where 1 is the worst candidate and $N+1$ is the best candidate. An interviewer is scheduled to interview the candidates one-by-one, and the order in which the interviewer interviews the candidates is random. The interviewer rejects the first m candidates, then begins interviewing the remaining candidates one-by-one. The interviewer then chooses the first person that is ranked higher than all of the m candidates that they rejected earlier. (This may not be possible, in which a secretary is simply not chosen.)
I claim that both sides of the equality calculate the probability that the candidate that is chosen is the highest ranked candidate.
The left hand side computes the probability, where we condition on the value of the highest rank among the first m candidates. In total, there are $\binom{N+1}{m}$ ways to choose the ranks of the first m candidates. Let $i+1$ be the highest rank among the first m candidates, where i ranges from $m-1$ to $N-1$. (The case that the highest rank among the first m candidates is $N+1$ contributes zero to the desired probability, so we can ignore that case.) The other $m-1$ candidates that were rejected must have their rank chosen from the range $1, 2, \ldots i$, so there are $\binom{i}{m-1}$ ways to choose their indices. Thus, the probability that the highest index is $i+1$ is $\frac{\binom{i}{m-1}}{\binom{N+1}{m}}$.
Now, for the highest ranked candidate to be chosen, the candidate that is ranked $N+1$ must appear before the candidates ranked $i+2, i+3, \ldots, N$. Since all orders of the candidates are equally likely, the probability of the highest ranked candidate appearing before $N-i-1$ other candidates is $\frac{1}{N-i}$.
Thus, the probability that the highest ranked candidate among the rejected candidates was $i+1$ and that the candidate ranked $N+1$ was chosen as secretary is $\frac{1}{\binom{N+1}{m}} \frac{\binom{i}{m-1}}{N-i}$. Summing over all i, we get that the probability that the highest candidate is picked is the left hand side.

We now show that the right hand side also computes the same probability; this time, we condition on the position of the $N+1$ ranked candidate.
Suppose that the $N+1$ ranked candidate will be the $i+1$th candidate interviewed, where i ranges from m to N. (Again, since the case that the $N+1$ ranked candidate is among the first m interviewed contributes zero to the desired probability, we can ignore that case.) First, there is a $\frac{1}{N+1}$ probability that the $N+1$ ranked candidate is the $i+1$th candidate interviewed.
Now, consider the ranks of the first i candidates. Let a be the highest rank among these candidates. For the $N+1$ ranked candidate to get selected, the candidate ranked a must be among the first m candidates rejected. This occurs with probability $\frac{m}{i}$. Thus, summing over all i gives that the desired probability is the right hand side.
We conclude

$$\sum_{m=1}^{N} a_{m,N} = \frac{1}{N+1}\sum_{m=1}^{N}\sum_{i=m}^{N}\frac{m}{i}.$$

However, we can swap the order of summation to get

$$\sum_{m=1}^{N}\sum_{i=m}^{N}\frac{m}{i} = \sum_{i=1}^{N}\sum_{m=1}^{i}\frac{m}{i} = \sum_{i=1}^{N}\frac{\frac{i(i+1)}{2}}{i}.$$

Simplifying, we get

$$\sum_{i=1}^{N}\frac{i+1}{2} = \frac{N(N+1)}{4} + \frac{N}{2} = \frac{N(N+3)}{4},$$

so

$$\sum_{m=1}^{N} a_{m,N} = \frac{N(N+3)}{4(N+1)} > \frac{2020N}{N+1}.$$

We simplify to get that $N+3 > 8080$, so $N > 8077$. Thus, the smallest positive integer N that works is $N = \boxed{8078}$.

25. Submit an integer between 1 and 50, inclusive. You will receive a score as follows:

If some number is submitted exactly once: If E is your number, A is the closest number to E which received exactly one submission, and M is the largest unique submission, you will receive $\frac{25}{M}(A - |E - A|)$ points, rounded to the nearest integer.

If no number was submitted exactly once: If E is your number, K is the number of people who submitted E, and M is the number of people who submitted the most commonly submitted number, then you will receive $\frac{25(M-K)}{M}$ points, rounded to the nearest integer.

Answer: N/A

Solution: We hope that you enjoyed the guts round!

26. Estimate the value of the 2020th prime number p such that $p+2$ is also prime. If $E > 0$ is your estimate and A is the correct answer, you will receive $25\min\left(\frac{E}{A}, \frac{A}{E}\right)^2$ points, rounded to the nearest integer. (An estimate less than or equal to 0 will receive 0 points.)

Answer: 183761

Solution: Among the first n positive integers, approximately $\frac{1}{\ln(n)}$ of the integers are primes. Thus, we estimate the probability that p and $p+2$ are both prime to be $\frac{1}{(\ln(n))^2}$. By Linearity of Expectation, the expected number of twin primes pairs among the first n positive integers is then $\frac{n}{(\ln(n))^2}$, so we want that $\frac{n}{(\ln(n))^2} \approx 2020$. This gives an estimate $n \approx 325444$, which would receive 8 points, due to a lot of error in our estimate; reducing the error would provide more points.

27. Estimate the number of 1s in the hexadecimal representation of 2020!. If E is your estimate and A is the correct answer, you will receive $\max(25 - 0.5|A - E|, 0)$ points, rounded to the nearest integer.

Answer: 277

Solution: By Stirling's approximation, we have that $\log_2(2020!) \approx 2020 \log_2 2020 - 2020/\ln(2)$.

$$\ln(2) \approx 0.693 \approx 0.7 * 0.99$$

$$\log_2(2020) \approx 11$$

$$\log_2(2020!) \approx 2020 \log_2 2020 - 2020/\ln(2) = 2020 * 11 - 2917 = 22220 - 2917 = 19303$$

Thus, the approximate number of hex digits in 2020! is about $19303/4 = 4826$. The total number of powers of 2 in 2020! is $1010 + 505 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2013$. We thus have the first $2013/4 = 503$ digits as zero, and the 504th digit as non-one. We thus have $4826 - 504 = 4322$ digits which can be 1. About $1/16$ of these should be 1, so we expect about $4322/16 = 270$ digits to be 1. This estimate would receive 22 points.