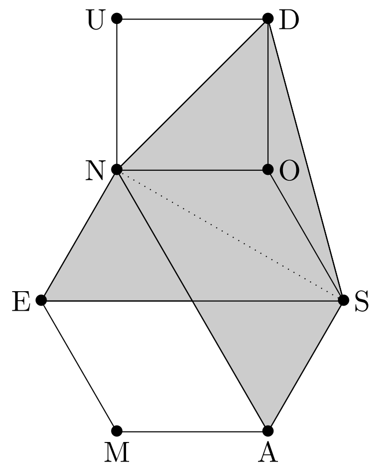
1. Regular hexagon $NOSAME$ with side length 1 and square $UDON$ are drawn in the plane such that $UDON$ lies outside of $NOSAME$. Compute $[SAND] + [SEND]$, the sum of the areas of quadrilaterals $SAND$ and $SEND$.

   Answer: $\frac{3+3\sqrt{3}}{2}$

   Solution:



We know $[SAND] = [SAN] + [SDN] = [SEN] + [SDN] = [SEND]$. So, our answer is $2 \cdot [SAND] = 2 \cdot ([SANO] + [DNO] + [DOS])$. We know $[DNO] = \frac{1}{2}$. We can calculate $[SANO]$ and $[SOD]$ by calculating the length of the altitudes altitudes from S to OA and NA to be $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$, respectively. So, our desired sum becomes $2 \cdot (\frac{3\sqrt{3}}{4} + \frac{1}{2} + \frac{1}{4}) = \boxed{\frac{3+3\sqrt{3}}{2}}$

2. Let $\triangle A_0B_0C_0$ be an equilateral triangle with area 1, and let A_1, B_1, C_1 be the midpoints of $\overline{A_0B_0}$, $\overline{B_0C_0}$, and $\overline{C_0A_0}$, respectively. Furthermore, set A_2, B_2, C_2 as the midpoints of segments $\overline{A_0A_1}$, $\overline{B_0B_1}$, and $\overline{C_0C_1}$ respectively. For $n \geq 1$, A_{2n+1} is recursively defined as the midpoint of $\overline{A_{2n}A_{2n-1}}$, and A_{2n+2} is recursively defined as the midpoint of $\overline{A_{2n+1}A_{2n-1}}$. Recursively define B_n and C_n the same way. Compute the value of $\lim_{n\to\infty}[A_nB_nC_n]$, where $[A_nB_nC_n]$ denotes the area of triangle $\triangle A_nB_nC_n$.

   Answer: $\frac{7}{25}$

   Solution: In order to find this area, we first find the limit of the ratio $r = \dfrac{A_0A_n}{A_0B_0}$. Observe that

$$\begin{aligned}\lim_{n\to\infty} r &= \frac{1}{A_0B_0}\left(A_0B_0 - A_1B_0 - A_2A_1 + A_3A_2 + A_4A_3 - \cdots\right)\\ &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} - \frac{1}{64} + \frac{1}{128} + \frac{1}{256} - \cdots = \frac{2}{5},\end{aligned}$$

so we have $\lim_{n\to\infty}[A_nB_nC_n] = 1 - 3\left(\lim_{n\to\infty} r\right)\left(1 - \lim_{n\to\infty} r\right)[A_0B_0C_0] = 1 - 3 \cdot \frac{2}{5} \cdot \frac{3}{5} = \boxed{\frac{7}{25}}$ by symmetry.

3. Right triangle $\triangle ABC$ with its right angle at B has angle bisector $\overline{AD}$ with D on $\overline{BC}$, as well as altitude $\overline{BE}$ with E on $\overline{AC}$. If $\overline{DE} \perp \overline{BC}$ and $AB = 1$, compute AC.

Answer: $\frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}$

Solution: Note that $\angle BAD = \angle ADE$ because of parallel lines. Since $\angle BAD = \angle DAE$, $\triangle ADE$ is isosceles. Let $AE = DE = x$ and $\angle BAC = \theta$. Then $\triangle BAE$ gives us $BE = x\tan\theta$ and $\triangle BED$ gives us $BE = \frac{x}{\sin\theta}$ since $\angle EBD = \angle BAC$. Thus, $\sin\theta \cdot \tan\theta = 1$, so we simplify to $\sin^2\theta = \cos\theta$. Plugging $\sin^2\theta = 1 - \cos^2\theta$ gives us $\cos^2\theta + \cos\theta - 1 = 0$. Now we can solve the quadratic to get $\cos\theta = \frac{-1+\sqrt{5}}{2}$, discarding the negative solution as $\angle BAC$ is acute. Thus, $AC = \boxed{\dfrac{1+\sqrt{5}}{2}}$.