1. Compute

$$\lim_{n\to\infty}\int_1^n \frac{\ln(x)}{n\ln(n)}\mathrm{d}x.$$

Answer: 1

Solution: Let ln denote the natural logarithm. To begin, we write

$$\lim_{n\to\infty}\int_1^n \frac{\ln(x)}{n\ln(n)}dx = \lim_{n\to\infty}\frac{\int_1^n \ln(x)dx}{n\ln(n)}.$$

Applying L'Hôpital's rule, this is

$$\begin{aligned}\lim_{n\to\infty}\frac{\int_1^n \ln(x)dx}{n\ln(n)} &= \lim_{n\to\infty}\frac{\ln(n)}{\ln(n)+n\cdot\frac{1}{n}}\\ &= \lim_{n\to\infty}\frac{\ln(n)}{\ln(n)+1}\\ &= \boxed{1}.\end{aligned}$$

2. Let $f(x) = e^x\sin(x)$. Compute $f^{(2022)}(0)$. Here, $f^{(2022)}(x)$ is the 2022nd derivative of $f(x)$.

Answer: -2^{1011}

Solution: The key observation is that

$$\begin{aligned}f(x) &= e^x\sin(x)\\ &= e^x\cdot\frac{e^{ix}-e^{-ix}}{2i}\\ &= \frac{e^{(1+i)x}-e^{(1-i)x}}{2i}.\end{aligned}$$

Taking iterated derivatives of the exponential, we find

$$f^{(2022)}(x) = \frac{(1+i)^{2022}e^{(1+i)x}-(1-i)^{2022}e^{(1-i)x}}{2i},$$

so

$$\begin{aligned}f^{(2022)}(0) &= \frac{(1+i)^{2022}-(1-i)^{2022}}{2i}\\ &= \mathrm{Im}\left((1+i)^{2022}\right)\\ &= \mathrm{Im}\left(2^{1011}e^{i3\pi/2}\right)\\ &= \boxed{-2^{1011}}.\end{aligned}$$

3. Compute

$$\int_{1/e}^{e}\frac{\arctan(x)}{x}\,\mathrm{d}x.$$

Answer: $\frac{\pi}{2}$

Solution: Set $x = e^u$ so that $\mathrm{d}x = e^u \mathrm{d}u$, which gives

$$\int_{1/e}^{e} \frac{\arctan(x)}{x}\,\mathrm{d}x = \int_{-1}^{1} \arctan\left(e^u\right)\mathrm{d}u.$$

Let the value of this integral be I. Now, if we set $v = -u$ so that $\mathrm{d}v = -\mathrm{d}u$, then we see

$$\begin{aligned} I &= \int_{1}^{-1} \arctan\left(e^{-v}\right) \cdot -\mathrm{d}v \\ &= \int_{-1}^{1} \arctan\left(e^{-v}\right)\mathrm{d}v \\ &= \int_{-1}^{1} \arctan\left(e^{-u}\right)\mathrm{d}u. \end{aligned}$$

However, we note that $\arctan(x) + \arctan(1/x) = \frac{\pi}{2}$ for any real $x > 0$, so

$$\begin{aligned} 2I &= \int_{-1}^{1} \left(\arctan\left(e^u\right) + \arctan\left(e^{-u}\right)\right)\mathrm{d}u \\ &= \int_{-1}^{1} \frac{\pi}{2}\mathrm{d}u \\ &= 2 \cdot \frac{\pi}{2}, \end{aligned}$$

so $I = \boxed{\frac{\pi}{2}}$.