Guts Round Scoring System

The format for the Guts Round is different from the other tests. The Guts Round lasts 75 minutes, and is divided into 9 sets of questions. You will start with Set 1, and will receive Set $k+1$ after submitting Set k. You can take as much time as you wish for each set, but you cannot go back to a previous set after submitting it.

Sets 1 through 8 consist of short answer questions, similar to the earlier tests. Either your answer is correct or incorrect, and points are given accordingly.

Set 9 (the last set) consists of estimation questions. Your score is based on how close your answer is to the correct answer. More details about scoring are given in Set 9.

Each question within the set is equally weighted, but each set is weighted differently.

- Set 1: 10 points per question
- Set 2: 11 points per question
- Set 3: 12 points per question
- Set 4: 13 points per question
- Set 5: 14 points per question
- Set 6: 16 points per question
- Set 7: 18 points per question
- Set 8: 20 points per question
- Set 9: 25 points per question

This is set 1.

1. What is the sum of all positive 2-digit integers whose sum of digits is 16?

2. A bag has 3 white and 7 black marbles. Arjun picks out one marble without replacement and then a second. What is the probability that Arjun chooses exactly 1 white and 1 black marble?

3. The polynomial $ax^2 + bx + c$ crosses the x-axis at $x = 10$ and $x = -6$ and crosses the y-axis at $y = 10$. Compute $a + b + c$.

This is set 2.

4. Compute the number of primes less than 40 that are the sum of two primes.

5. Theo and Wendy are commuting to school from their houses. Theo travels at x miles per hour, while Wendy travels at $x + 5$ miles per hour. The school is 4 miles from Theo's house and 10 miles from Wendy's house. If Wendy's commute takes double the amount of time that Theo's commute takes, how many *minutes* does it take Wendy to get to school?

6. Equilateral triangle ABC has side length 20. Let $PQRS$ be a square such that A is the midpoint of $\overline{RS}$ and Q is the midpoint of $\overline{BC}$. Compute the area of $PQRS$.

This is set 3.

7. A regular hexagon is inscribed in a circle of radius 1, and all diagonals between vertices that have exactly one vertex between them are drawn. Compute the area of the hexagon enclosed by all of the diagonals.

8. Seven equally-spaced points are drawn on a circle of radius 1. Three distinct points are chosen uniformly at random. What is the probability that the center of the circle lies in the triangle formed by the three points?

9. Define the polynomial $f(x) = x^4 + x^3 + x^2 + x + 1$. Compute the number of positive integers n less than equal to 2022 such that $f(n)$ is 1 more than multiple of 5.

This is set 4.

10. Compute the number of ordered pairs (a, b) of positive integers such that a and b divide 5040 but share no common factors greater than 1.

11. Kylie is trying to count to 202250. However, this would take way too long, so she decides to only write down positive integers from 1 to 202250, inclusive, that are divisible by 125. How many times does she write down the digit 2?

12. Let circles C_1 and C_2 be internally tangent at point P, with C_1 being the smaller circle. Consider a line passing through P which intersects C_1 at Q and C_2 at R. Let the line tangent to C_2 at R and the line perpendicular to $\overline{PR}$ passing through Q intersect at a point S outside both circles. Given that $SR = 5$, $RQ = 3$, and $QP = 2$, compute the radius of C_2.

This is set 5.

13. Real numbers x and y satisfy the system of equations

$$x^3 + 3x^2 = -3y - 1$$
$$y^3 + 3y^2 = -3x - 1.$$

What is the greatest possible value of x?

14. Isaac writes each fraction $\frac{1^2}{300}, \frac{2^2}{300}, \dots, \frac{300^2}{300}$ in reduced form. Compute the sum of all denominators over all the reduced fractions that Isaac writes down.

15. Let $f(x)$ be a function acting on a string of 0s and 1s, defined to be the number of substrings of x that have at least one 1, where a substring is a contiguous sequence of characters in x. Let S be the set of binary strings with 24 ones and 100 total digits. Compute the maximum possible value of $f(s)$ over all $s \in S$.

For example, $f(110) = 5$ as $\underline{1}10, 1\underline{1}0, \underline{11}0, 1\underline{10}$, and $\underline{110}$ are all substrings including a 1. Note that $11\underline{0}$ is not such a substring.

This is set 6.

16. Let triangle $\triangle ABC$ be a triangle with $AB = 5$, $BC = 7$, and $CA = 8$, and let I be the incenter of $\triangle ABC$. Let circle C_A denote the circle with center A and radius $\overline{AI}$, denote C_B and circle C_C similarly. Besides all intersecting at I, the circles C_A, C_B, C_C also intersect pairwise at F, G, and H. Compute the area of triangle $\triangle FGH$.

17. Compute the number of ordered triples (a, b, c) of integers between -100 and 100 inclusive satisfying the simultaneous equations

$$a^3 - 2a = abc - b - c$$
$$b^3 - 2b = bca - c - a$$
$$c^3 - 2c = cab - a - b.$$

18. Nir finds integers $a_0, a_1, \ldots, a_{208}$ such that

$$(x+2)^{208} = a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_{208}x^{208}.$$

Let S be the sum of all a_n such that $n - 3$ is divisible by 5. Compute the remainder when S is divided by 103.

This is set 7.

19. Let $N \geq 3$ be the answer to Problem 21. A regular N-gon is inscribed in a circle of radius 1. Let D be the set of diagonals, where we include all sides as diagonals. Then, let D' be the set of all unordered pairs of distinct diagonals in D. Compute the sum

$$\sum_{\{d,d'\} \in D'} \ell(d)^2 \ell(d')^2,$$

where $\ell(d)$ denotes the length of diagonal d.

20. Let N be the answer to Problem 19, and let M be the last digit of N. Let ω be a primitive Mth root of unity, and define $P(x)$ such that

$$P(x) = \prod_{k=1}^{M} \left(x - \omega^{i_k}\right),$$

where the i_k are chosen independently and uniformly at random from the range $\{0, 1, \dots, M-1\}$. Compute $\mathbb{E}\left[P\left(\sqrt{\lfloor \frac{1250}{N} \rfloor}\right)\right]$.

21. Let N be the answer to Problem 20. Define the polynomial $f(x) = x^{34} + x^{33} + x^{32} + \cdots + x + 1$. Compute the number of primes $p < N$ such that there exists an integer k with $f(k)$ divisible by p.

This is set 8.

22. Set $n = 425425$. Let S be the set of proper divisors of n. Compute the remainder when

$$\sum_{k \in S} \varphi(k) \binom{2n/k}{n/k}$$

is divided by $2n$, where $\varphi(x)$ is the number of positive integers at most x that are relatively prime to it.

23. Carson the farmer has a plot of land full of crops in the shape of a 6×6 grid of squares. Each day, he uniformly at random chooses a row or a column of the plot that he hasn't chosen before and harvests all of the remaining crops in the row or column. Compute the expected number of connected components that the remaining crops form after 6 days. If all crops have been harvested, we say there are 0 connected components.

24. Let $\triangle BCD$ be an equilateral triangle and A be a point on the circumcircle of $\triangle BCD$ such that A is on the minor arc $\widehat{BD}$. Then, let P be the intersection of $\overline{AB}$ with $\overline{CD}$, Q be the intersection of $\overline{AC}$ with $\overline{DB}$, and R be the intersection of $\overline{AD}$ with $\overline{BC}$. Finally, let X, Y, and Z be the feet of the altitudes from P, Q, and R, respectively, in triangle $\triangle PQR$. Given $BQ = 3 - \sqrt{5}$ and $BC = 2$, compute the product of the areas $[\triangle XCD] \cdot [\triangle YDB] \cdot [\triangle ZBC]$.

This is set 9.

25. For triangle $\triangle ABC$, define its *A-excircle* to be the circle that is externally tangent to line segment $\overline{BC}$ and extensions of $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, and define the B-excircle and C-excircle likewise. Then, define the *A-veryexcircle* to be the unique circle externally tangent to both the A-excircle as well as the extensions of $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, but that shares no points with line $\overleftrightarrow{BC}$, and define the B-veryexcircle and C-veryexcircle likewise.

    Compute the smallest integer $N \geq 337$ such that for all $N_1 \geq N$, the area of a triangle with lengths $3N_1^2$, $3N_1^2 + 1$, and $2022N_1$ is at most $\frac{1}{22022}$ times the area of the triangle formed by connecting the centers of its three veryexcircles. If your submitted estimate is a positive number E and the true value is A, then your score is given by $\max\left(0, \left\lfloor 25\min\left(\frac{E}{A}, \frac{A}{E}\right)^3\right\rfloor\right)$.

26. Compute the number of positive integers n less than 10^8 such that at least two of the last five digits of

$$\left\lfloor 1000\sqrt{25n^2 + \frac{50}{9}n + 2022}\right\rfloor$$

    are 6. If your submitted estimate is a positive number E and the true value is A, then your score is given by $\max\left(0, \left\lfloor 25\min\left(\frac{E}{A}, \frac{A}{E}\right)^7\right\rfloor\right)$.

27. Submit a positive integer n less than 10^5. Let the sum of the valid submissions from all teams to this question be S. If you submit an invalid answer, you will receive 0 points. Otherwise, your score will be $\max\left(0, \left\lfloor 25 - \frac{|S'-n|}{10}\right\rfloor\right)$, where S' is the sum of the squares of the digits of S.