1. What is the sum of all positive 2-digit integers whose sum of digits is 16?

   Answer: 264

   Solution: Notice that all digits must be less than 10. Therefore, the second digit must be at most 9, so the first digit must be at least 7. Moreover, the first digit must be at most 9. So, we can see that the answer is $79 + 88 + 97 = \boxed{264}$.

2. A bag has 3 white and 7 black marbles. Arjun picks out one marble without replacement and then a second. What is the probability that Arjun chooses exactly 1 white and 1 black marble?

   Answer: $\mathbf{\frac{7}{15}}$

   Solution: The probability that Arjun picks exactly 1 white and 1 black marble is the sum of the probability that the first marble picked is white and the second marble is black, and the probability that the first marble picked is black and the second marble is white. However, by symmetry, the probability of the first case should equal to the probability of the second case. The probability of the first case is $\frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}$, so our answer is $2 \cdot \frac{7}{30} = \boxed{\frac{7}{15}}$.

3. The polynomial $ax^2 + bx + c$ crosses the x-axis at $x = 10$ and $x = -6$ and crosses the y-axis at $y = 10$. Compute $a + b + c$.

   Answer: $\mathbf{\frac{21}{2}}$

   Solution: Our polynomial has x-intercepts of 10 and -6, which means that it can be factored as $A(x-10)(x+6) = A\left(x^2 - 4x - 60\right)$, where A is some constant. The y-intercept tells us that $10 = A(0 + 0 - 60)$ so $A = -\frac{1}{6}$. Thus, $a + b + c = -\frac{1}{6}(1 - 4 - 60) = \boxed{\frac{21}{2}}$.

4. Compute the number of primes less than 40 that are the sum of two primes.

   Answer: 5

   Solution: If a prime p is the sum of two primes, q and r, then $p = q + r \geq 2 + 2 > 2$, so p is odd (because 2 is the only even prime). But then one of q or r must be even. Because the only even prime is 2, take $r = 2$. This means that $p = q + 2$, so p must be the greater of a pair of twin primes. Listing the twin primes less than 40, we get:

   $$(3,5), \quad (5,7), \quad (11,13), \quad (17,19), \quad (29,31).$$

   This totals to $\boxed{5}$ total solutions for p.

5. Theo and Wendy are commuting to school from their houses. Theo travels at x miles per hour, while Wendy travels at $x + 5$ miles per hour. The school is 4 miles from Theo's house and 10 miles from Wendy's house. If Wendy's commute takes double the amount of time that Theo's commute takes, how many *minutes* does it take Wendy to get to school?

   Answer: 24

   Solution: Note that we can divide both sides of the $d = rt$ formula by r to get $t = \frac{d}{r}$. Therefore, Theo's commute is $\frac{4}{x}$ hours, and Wendy's commute is $\frac{10}{x+5}$ hours. Because Wendy's commute time is double Theo's, we compute

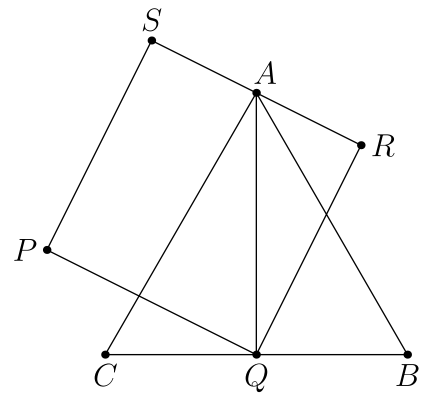
   $$\begin{aligned} \frac{10}{x+5} &= 2 \cdot \frac{4}{x} \\ 10x &= 8(x+5) \\ x &= 20. \end{aligned}$$

Therefore, Wendy's trip takes $\frac{10}{x+5} = \frac{10}{25} = \frac{2}{5}$ hours, which is $\frac{2}{5} \cdot 60 = \boxed{24}$ minutes.

6. Equilateral triangle ABC has side length 20. Let $PQRS$ be a square such that A is the midpoint of $\overline{RS}$ and Q is the midpoint of $\overline{BC}$. Compute the area of $PQRS$.

Answer: 240

Solution:



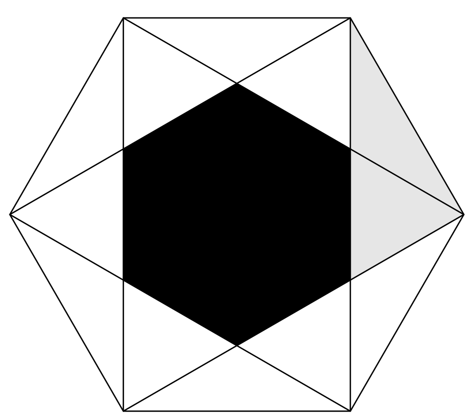
Let the side length of $PQRS$ be x, so that our answer is x^2. Observe that $\triangle ARQ$ is a right triangle with hypotenuse $AQ = \frac{\sqrt{3}}{2} \cdot 20 = 10\sqrt{3}$ and legs with lengths $\frac{x}{2}$ and x. Thus,

$$\left(\frac{x}{2}\right)^2 + x^2 = \left(10\sqrt{3}\right)^2$$
$$\frac{5}{4} \cdot x^2 = 300$$
$$x^2 = \boxed{240}.$$

7. A regular hexagon is inscribed in a circle of radius 1, and all diagonals between vertices that have exactly one vertex between them are drawn. Compute the area of the hexagon enclosed by all of the diagonals.

Answer: $\frac{\sqrt{3}}{2}$

Solution:



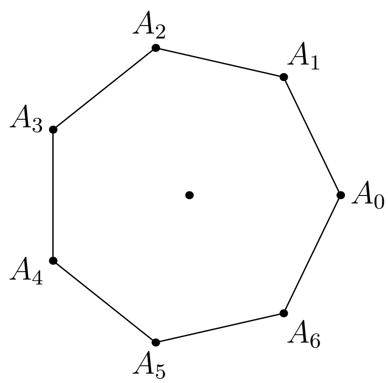
We want to compute the area of the black hexagon, or equivalently the area of the large hexagon minus the area of 6 of the gray right triangles. The large hexagon has side length 1, so it consists of 6 equilateral triangles with unit side length. So, the area of the hexagon is $\frac{3\sqrt{3}}{2}$, and each gray right triangle has area $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}}$. The desired area is

$$\frac{3\sqrt{3}}{2} - 6 \cdot \frac{1}{2\sqrt{3}} = \boxed{\frac{\sqrt{3}}{2}}.$$

8. Seven equally-spaced points are drawn on a circle of radius 1. Three distinct points are chosen uniformly at random. What is the probability that the center of the circle lies in the triangle formed by the three points?

Answer: $\mathbf{\frac{2}{5}}$

Solution: Here is our diagram.



Without loss of generality, we may assume that one of the chosen points is A_0. Notice that if we choose none of $\{A_1, A_2, A_3\}$, then there is no way to contain the center as the triangle will always fall on the wrong side of line $\overline{A_0A_4}$. We now proceed with casework.

- If A_1 is chosen, then the only point which will give a triangle containing the origin is A_4.
- If A_2 is chosen, then the only points which will give a triangle containing the origin are those in $\{A_4, A_5\}$.
- If A_3 is chosen, then the only points which will give a triangle containing the origin are those in $\{A_4, A_5, A_6\}$.

In total, there we found 6 possible triangles, out of a total of $\binom{6}{2} = 15$ ways to choose two points given A_0, so the probability is $\frac{6}{15} = \boxed{\frac{2}{5}}$.

9. Define the polynomial $f(x) = x^4 + x^3 + x^2 + x + 1$. Compute the number of positive integers n less than equal to 2022 such that $f(n)$ is 1 more than multiple of 5.

Answer: 1617

Solution: If $x \equiv 1 \pmod 5$, then $f(x) \equiv 0 \pmod 5$, which fails. If $x \not\equiv 1 \pmod 5$, then by

Fermat's little theorem,

$$\begin{aligned} x^4 + x^3 + x^2 + x + 1 &= \frac{x^5 - 1}{x - 1} \\ &\equiv \frac{x - 1}{x - 1} \\ &\equiv 1 \pmod 5. \end{aligned}$$

Thus, the number of integers satisfying the equation is the number of integers in the range $1, \ldots, 2022$ that are not 1 modulo 5, which gives us an answer of $2022 - \left(\frac{2020}{5} + 1\right) = \boxed{1617}$.

10. Compute the number of ordered pairs (a, b) of positive integers such that a and b divide 5040 but share no common factors greater than 1.

Answer: 405

Solution: Set $N = 5040 = 7!$, and factor N as $N = 2^4 \cdot 3^2 \cdot 5 \cdot 7$. For convenience, for given prime p and positive integer n, we will let $\nu_p(n)$ denote the largest power of p dividing n.

Now, consider the prime factorizations of a and b, which uniquely determine the ordered pair (a, b). We can only use the primes p dividing N, and for each $p \mid N$, we note that there are $2\nu_p(N) + 1$ total options for the ordered pair $(\nu_p(a), \nu_p(b))$: we can have p divide neither a nor b, we can have a positive power of p in $\left\{p, p^2, \ldots, p^{\nu_p(N)}\right\}$ divide a and not b, or we can have a positive power of p divide b and not a. Importantly, a positive power of p cannot divide both.

Thus, our answer is

$$\prod_{p \mid N} (2\nu_p(N) + 1) = \underbrace{(2 \cdot 4 + 1)}_{p=2} \underbrace{(2 \cdot 2 + 1)}_{p=3} \underbrace{(2 \cdot 1 + 1)}_{p=5} \underbrace{(2 \cdot 1 + 1)}_{p=7} = \boxed{405}.$$

11. Kylie is trying to count to 202250. However, this would take way too long, so she decides to only write down positive integers from 1 to 202250, inclusive, that are divisible by 125. How many times does she write down the digit 2?

Answer: 950

Solution: We count by digit.

- Hundred thousands: there are $\frac{202250 - 200000}{125} + 1 = 19$ numbers whose hundred thousands digit is a two.
- Ten-thousands and thousands: there are $\frac{200000}{125} = 1600$ multiples of 125 between 1 and 200000. Here, all the digits are represented equally, there are $\frac{1}{10} \cdot 1600 = 160$ written twos for each of the thousands and ten-thousands digit.

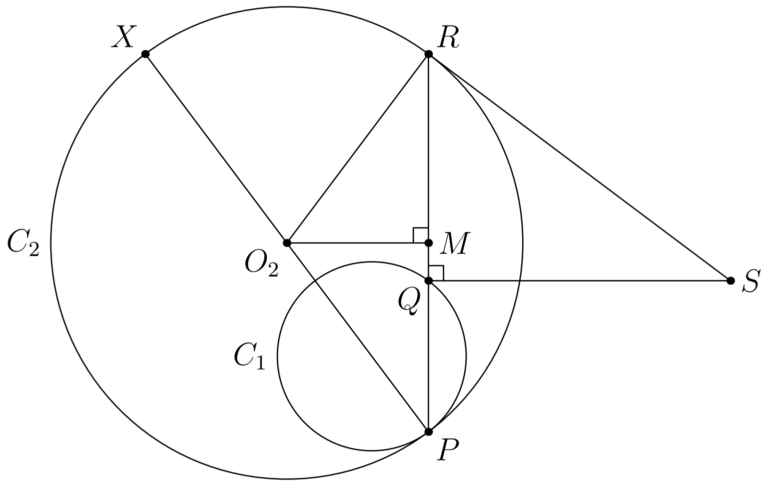
  Additionally, there are 3 instances of two for the thousands digit and no instances of two in the ten-thousands digit for multiples of 125 between 200125 to 202250.
- Hundreds: the hundreds digit cycles through 0, 1, 2, 3, 5, 6, 7, and 8, so there are $\frac{1600}{8} + 3 = 203$ instances of two for the hundreds digit.
- Tens: the tens digit alternates between 0, 2, 5, and 7, so there are $\frac{1600}{4} + 5 = 405$ instances of two.
- Ones: The ones digit alternates between 0 and 5, so there are no twos in the ones digit.

Summing, the total is $19 + 2 \cdot 160 + 3 + 203 + 405 = \boxed{950}$.

12. Let circles C_1 and C_2 be internally tangent at point P, with C_1 being the smaller circle. Consider a line passing through P which intersects C_1 at Q and C_2 at R. Let the line tangent to C_2 at R and the line perpendicular to $\overline{PR}$ passing through Q intersect at a point S outside both circles. Given that $SR = 5$, $RQ = 3$, and $QP = 2$, compute the radius of C_2.

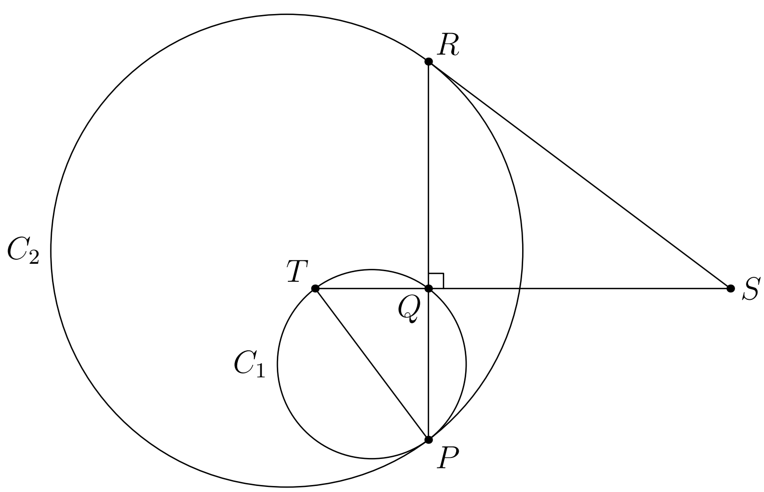
Answer: $\mathbf{\frac{25}{8}}$

Solution 1:



Draw diameter $\overline{PX}$ of C_2. Note that $SQ = 4$ by the Pythagorean theorem. Let M be the midpoint of $\overline{PR}$. Note that the line through M perpendicular to $\overline{PR}$ intersects $\overline{PX}$ at O_2, the center of circle C_2. Because $\angle O_2RS = 90^\circ$ by properties of the tangent line, then $\triangle SRQ$ is similar to $\triangle RO_2M$. Thus, $\frac{O_2R}{MR} = \frac{RS}{QS} = \frac{5}{4}$ while $MR = \frac{PR}{2} = \frac{5}{2}$, so $O_2R = \boxed{\frac{25}{8}}$.

Solution 2:



Let the line through Q perpendicular to PR intersect C_1 at T. We know $\angle SRQ$ is equal to the angle intercepted by arc $\widehat{PR}$ on C_2. Because C_1 and C_2 are homothetic about point P, this is also equal to $\angle PTQ$. So, $\triangle SRQ \sim \triangle PTQ$. By the Pythagorean theorem, $SQ = 4$, so we get that $PT = \frac{PQ}{SQ} \cdot SR = \frac{5}{2}$. Because $\angle PQT = 90^\circ$, we know PT is the diameter of C_1. We can multiply by $\frac{PQ}{PR} = \frac{5}{2}$ to get the diameter of C_2 to be $\frac{25}{4}$, so our answer is $\boxed{\frac{25}{8}}$.

13. Real numbers x and y satisfy the system of equations

$$x^3 + 3x^2 = -3y - 1$$
$$y^3 + 3y^2 = -3x - 1.$$

What is the greatest possible value of x?

Answer: $\sqrt{6} - 1$

Solution: These equations look a lot like $(x + 1)^3$ and $(y + 1)^3$ except for a mismatched linear term. Rearranging these equations gives

$$(x + 1)^3 = 3(x - y)$$
$$(y + 1)^3 = 3(y - x).$$

At this point, we are hinted towards setting $a = x + 1$ and $b = y + 1$, which makes our system of equations into

$$a^3 = 3(a - b)$$
$$b^3 = 3(b - a).$$

In particular, $a^3 = -b^3$, so because these expression are real, $a = -b$. Substituting, we see

$$a^3 = 3(a - b) = 6a,$$

so $a \in \{-\sqrt{6}, 0, \sqrt{6}\}$. We can check that each value of a does in fact give a valid pair of real numbers (a, b) satisfying the system. Thus, the largest possible value for x is $x = a - 1 = \boxed{\sqrt{6} - 1}$.

14. Isaac writes each fraction $\frac{1^2}{300}, \frac{2^2}{300}, \ldots, \frac{300^2}{300}$ in reduced form. Compute the sum of all denominators over all the reduced fractions that Isaac writes down.

Answer: 35350

Solution: Let $n = 300$. The fraction $\frac{k^2}{n}$ reduces as

$$\frac{k^2/\gcd\left(n, k^2\right)}{n/\gcd\left(n, k^2\right)},$$

so we must compute

$$f(n) := \sum_{k=1}^{n} \frac{n}{\gcd\left(n, k^2\right)}.$$

We claim that f is multiplicative over relatively prime numbers. Indeed, if $\gcd(m_1, m_2) = 1$, then any divisor $d \mid m_1m_2$ can be decomposed into $d = \gcd(d, m_1) \cdot \gcd(d, m_2)$. Thus, we can write

$$\begin{aligned} f(m_1m_2) &= \sum_{k=1}^{m_1m_2} \frac{m_1m_2}{\gcd\left(m_1m_2, k^2\right)} \\ &= \sum_{k=1}^{m_1m_2} \left(\frac{m_1}{\gcd\left(m_1, k^2\right)} \cdot \frac{m_2}{\gcd\left(m_2, k^2\right)}\right) \\ &= \left(\sum_{k_1=1}^{m_1} \frac{m_1}{\gcd\left(m_1, k_1^2\right)}\right)\left(\sum_{k_2=1}^{m_2} \frac{m_2}{\gcd\left(m_2, k_2^2\right)}\right) \qquad (*) \\ &= f(m_1)f(m_2), \end{aligned}$$

where $(*)$ follows from the Chinese Remainder Theorem.

Finally, since $300 = 2^2 \cdot 3 \cdot 5^2$, we compute

$$\begin{aligned} f(300) &= f\left(2^2\right) f(3) f\left(5^2\right) \\ &= (4+1+4+1)(3+3+1)(20 \cdot 25 + 5 \cdot 1) \\ &= \boxed{35350}. \end{aligned}$$

15. Let $f(x)$ be a function acting on a string of 0s and 1s, defined to be the number of substrings of x that have at least one 1, where a substring is a contiguous sequence of characters in x. Let S be the set of binary strings with 24 ones and 100 total digits. Compute the maximum possible value of $f(s)$ over all $s \in S$.

For example, $f(110) = 5$ as $\underline{1}10, 1\underline{1}0, \underline{11}0, 1\underline{10}$, and $\underline{110}$ are all substrings including a 1. Note that $11\underline{0}$ is not such a substring.

Answer: 4896

Solution: Once we lay out the 24 ones, we have 25 gaps in which to place the remaining 76 zeros. Note that the number of subsequences that contain a 1 is equal to the total number of subsequences minus the number of subsequences that do not contain a one, so we will minimize this second quantity.

The number of subsequences that don't contain one is the sum of the number of subsequences of all zeros that can be formed inside one of the 25 "gaps" of zeros in the string. Because a string of length n has $\binom{n+1}{2}$ total sequences, we want to minimize

$$\sum_{i=1}^{25} \binom{a_i + 1}{2},$$

where a_i is the number of zeros in the ith gap between consecutive ones. Note that our only other constraint on the a_i besides being nonnegative is that $a_1 + a_2 + \cdots + a_{25} = 76$.

We claim that it is best to make all of the a_i as close together as possible. Indeed, for all positive integers m, n with $m \le n$, we have

$$\begin{aligned} \binom{m}{2} + \binom{n}{2} &= \frac{m^2 - m + n^2 - n}{2} \\ &\le \frac{m^2 - 3m + n^2 + n}{2} \\ &< \frac{m^2 - 3m + 2 + n^2 + n}{2} \\ &= \binom{m-1}{2} + \binom{n+1}{2}, \end{aligned}$$

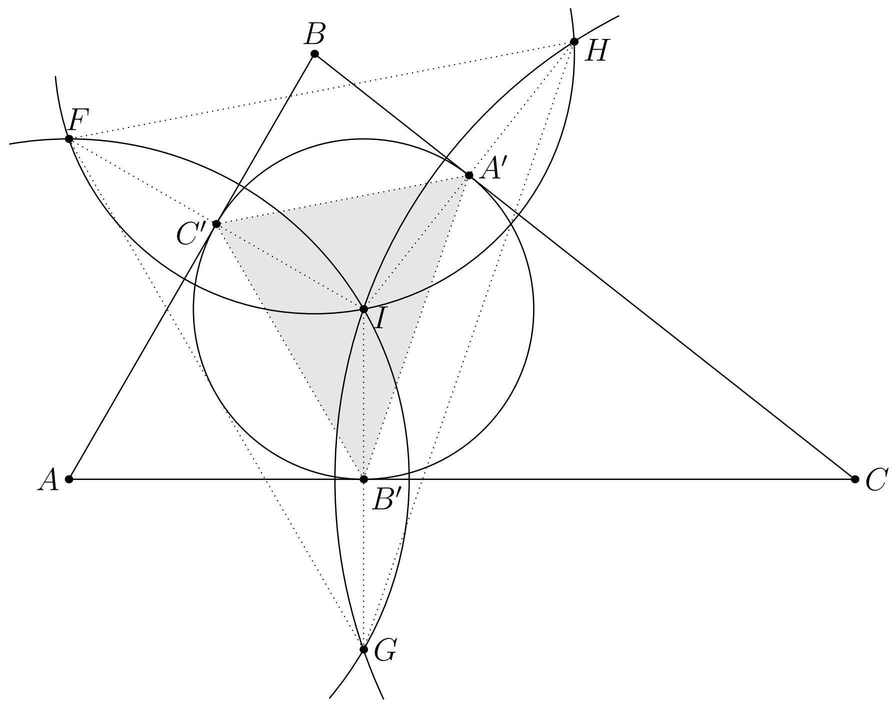
so $\binom{m}{2} + \binom{n}{2} < \binom{m-1}{2} + \binom{n+1}{2}$. Thus, it is always optimal to choose the a_i to be as close as we can. In our case, with 76 zeros and 25 gaps, we cannot make all the a_i equal, but we can choose 24 of the gaps to have length three and select the last gap to have length four. In total, our answer is $\binom{101}{2} - \left(24\binom{3+1}{2} + \binom{4+1}{2}\right) = 5050 - 154 = \boxed{4896}$.

16. Let triangle $\triangle ABC$ be a triangle with $AB = 5$, $BC = 7$, and $CA = 8$, and let I be the incenter of $\triangle ABC$. Let circle C_A denote the circle with center A and radius $\overline{AI}$, denote C_B and circle

C_C similarly. Besides all intersecting at I, the circles C_A, C_B, C_C also intersect pairwise at F, G, and H. Compute the area of triangle $\triangle FGH$.

Answer: $\frac{60\sqrt{3}}{7}$

Solution: We will build the following diagram.



Without loss of generality, let C_A and C_B intersect at F, C_A and C_C intersect at G, and C_B and C_C intersect at H.

Let the incircle be tangent to $\triangle ABC$ at A', B', C' where A' is on BC, B' on CA, and C' on AB. Then, A' is the intersection point of $\overline{BC}$ and $\overline{IH}$ and therefore bisects $\overline{IH}$, and same for B' and C', so $\triangle FGH$ is homothetic to $\triangle C'B'A'$ via a homothety centered on I with factor $\frac{1}{2}$. So, it becomes a matter of finding the area of triangle $\triangle A'B'C'$. Triangle $\triangle A'B'C'$ is the Gergonne triangle of $\triangle ABC$, so its area is

$$[\triangle A'B'C'] = [\triangle ABC]\frac{2r^2 s}{abc},$$

where a, b, and c are the lengths of $\triangle ABC$, s is the semiperimeter of $\triangle ABC$, and r is the inradius of $\triangle ABC$.

If one does not know this formula, one could find the area of the triangle a different way. Notice that the Gergonne triangle divides the triangle into four smaller triangles, so to find the area, we find the area of the larger triangle minus the area of the three other triangles.

For example, the area $[\triangle AB'C']$ is $\frac{1}{2}(s-a)^2\sin(A)$. But $[\triangle ABC] = \frac{1}{2}bc\sin(A)$, so $[\triangle AB'C'] = \frac{(s-a)^2}{bc}[\triangle ABC]$. So, subtracting out the three triangles,

$$[\triangle A'B'C'] = [\triangle ABC]\left(1 - \frac{(s-a)^2}{bc} - \frac{(s-b)^2}{ac} - \frac{(s-c)^2}{ab}\right).$$

We compute using Heron's formula and the formula $[\triangle ABC] = rs$ that $s = 10$ and $[\triangle ABC] = 10\sqrt{3}$ so $r = \sqrt{3}$. Thus,

$$[\triangle A'B'C'] = \frac{15\sqrt{3}}{7},$$

so the answer is $2^2 \cdot \frac{15\sqrt{3}}{7} = \boxed{\frac{60\sqrt{3}}{7}}$.

17. Compute the number of ordered triples (a, b, c) of integers between -100 and 100 inclusive satisfying the simultaneous equations

$$\begin{aligned} a^3 - 2a &= abc - b - c \\ b^3 - 2b &= bca - c - a \\ c^3 - 2c &= cab - a - b. \end{aligned}$$

Answer: 207

Solution: Adding the equations, we get

$$a^3 + b^3 + c^3 = 3abc,$$

which is

$$(a + b + c)\left((a - b)^2 + (b - c)^2 + (c - a)^2\right) = 0,$$

so either $a + b + c = 0$ or $a = b = c$. Plugging in $a = b = c$ to the equations yields equalities for all 3, so this case instantly gives us 201 triples.

Otherwise, we assume that $a + b + c = 0$. Subtracting a from both sides of the first equation gives

$$a^3 - 3a = abc - a - b - c = abc,$$

so $a = 0$ or $a^2 - 2 = bc$. But if $a = 0$, then the other two equations become $b^3 - 3b = c^3 - 3c = 0$, which forces $(a, b, c) = (0, 0, 0)$ because a, b, c are integers, which we've already accounted for. Thus, we must have the simultaneous equations

$$\begin{aligned} a^2 - 3 &= bc \\ b^2 - 3 &= ca \\ c^2 - 3 &= ab. \end{aligned}$$

Adding them gives us

$$a^2 + b^2 + c^2 - 9 = ab + ac + bc,$$

which, after doubling the equation, rearranges into

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 18.$$

Now, there are only two ways to write 18 as the sum of three integer squares: $0 + 9 + 9$ and $1 + 1 + 16$. Thus, $a - b, b - c, c - a$ are $0, \pm 3, \pm 3$ or $\pm 1, \pm 1, \pm 4$ in some order. However, because $(a - b) + (b - c) + (c - a) = 0$, our only potential solutions are $a - b, b - c, c - a$ equaling $0, 3, -3$ in some order.

Without loss of generality, we let $a - b = 0$ and $c - a = \pm 3$, so substituting into the second system of equations yields

$$\begin{aligned} a^2 - 3 &= a(a \pm 3), \\ (a \pm 3)^2 - 3 &= a^2. \end{aligned}$$

This gives $(a, b, c) = (\mp 1, \mp 1, \pm 2)$. These also satisfy our original equations, so the total amount of solutions is $201 + 3 \cdot 2 = \boxed{207}$.

18. Nir finds integers $a_0, a_1, \ldots, a_{208}$ such that

$$(x+2)^{208} = a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_{208}x^{208}.$$

Let S be the sum of all a_n such that $n - 3$ is divisible by 5. Compute the remainder when S is divided by 103.

Answer: 17

Solution: Let $p = 103$ so that $208 = 2p + 2$. The key is to use the fact that $(a + b)^p \equiv a^p + b^p \pmod p$ for any $a, b \in \mathbb{Z}[x]$. Now, we note that we might as well compute $(\text{mod } p, x^5 - 1)$ because we are only concerned with the value of n in x^n up to modulo 5. As such, we compute

$$\begin{aligned} (x+2)^{2p+2} &\equiv (x+2)^{2p}(x+2)^2 \\ &\equiv (x^p+2)^2\,(x+2)^2 \\ &\equiv \left(x^3+2\right)^2(x+2)^2 \\ &\equiv \left(x^4+2x^3+2x+4\right)^2 \quad (\text{mod } p, x^5-1). \end{aligned}$$

Now, the only terms with exponent $3 \pmod 5$ are $(2 \cdot 4 + 4 \cdot 2)x^3$ and $1x^8$, leaving us with $16 + 1 = \boxed{17}$ as our answer.

19. Let $N \geq 3$ be the answer to Problem 21. A regular N-gon is inscribed in a circle of radius 1. Let D be the set of diagonals, where we include all sides as diagonals. Then, let D' be the set of all unordered pairs of distinct diagonals in D. Compute the sum

$$\sum_{\{d,d'\} \in D'} \ell(d)^2 \ell(d')^2,$$

where $\ell(d)$ denotes the length of diagonal d.

Answer: 275

Solution: Embed the problem in the complex plane; let ζ be a primitive Nth root of unity so that the vertices of our regular N-gon are $\zeta^0, \zeta^1, \ldots, \zeta^{N-1}$. We are interested in computing

$$S = \frac{1}{8} \sum_{(a,b) \neq (c,d)} \left(\left| \zeta^a - \zeta^b \right| \cdot \left| \zeta^c - \zeta^d \right| \right)^2,$$

where the $\frac{1}{8}$ comes in because we are double-counting each diagonal and each pair of diagonals. (Terms with $a = b$ or $c = d$ contribute nothing and so can be safely ignored.) We split S in two pieces as

$$S = \underbrace{\frac{1}{8} \sum_{(a,b),(c,d)} \left(\left| \zeta^a - \zeta^b \right| \cdot \left| \zeta^c - \zeta^d \right| \right)^2}_{S_1} - \underbrace{\frac{1}{4} \sum_{(a,b)} \left(\left| \zeta^a - \zeta^b \right| \cdot \left| \zeta^a - \zeta^b \right| \right)^2}_{S_2}$$

and compute the pieces independently. Namely, the $\frac{1}{8}$ on the left is inherited from the previous sum, and the current $\frac{1}{4}$ comes from terms with $(a, b) = (c, d)$ and $a \neq b$ appearing twice in S_1.

- We compute S_1. Noting $|z| = z\overline{z}$, we see

$$S_1 = \sum_{(a,b),(c,d)} \Big((\zeta^a - \zeta^b)(\zeta^{-a} - \zeta^{-b})(\zeta^c - \zeta^d)(\zeta^{-c} - \zeta^{-d})\Big)$$
$$= \sum_{(a,b),(c,d)} \Big((2 - \zeta^{a-b} - \zeta^{b-a})(2 - \zeta^{c-d} - \zeta^{d-c})\Big).$$

  Expanding this, we get that S is simply:

$$\sum_{(a,b),(c,d)} \left(4 + \zeta^{a-b+c-d} + \zeta^{a-b+d-c} + \zeta^{b-a+c-d} + \zeta^{b-a+d-c} - 2\left(\zeta^{a-b} + \zeta^{b-a} + \zeta^{c-d} + \zeta^{d-c}\right)\right).$$

  Because $N > 2$, all the sums over nontrivial roots of unity must vanish, so we are left with

$$S_1 = \sum_{(a,b),(c,d)} 4 = 4N^4.$$

- We compute S_2. Again using that $|z|^2 = z\overline{z}$, this is

$$S_2 = \sum_{(a,b)} \Big((\zeta^a - \zeta^b)(\zeta^{-a} - \zeta^{-b})\Big)^2$$
$$= \sum_{(a,b)} \left(2 - \zeta^{a-b} - \zeta^{b-a}\right)^2$$
$$= \sum_{(a,b)} \left(4 + \zeta^{2a-2b} + \zeta^{2b-2a} - 4\zeta^{a-b} - 4\zeta^{b-a} + 2\right).$$

  Once more, we see that the sums over ζ^{2a-2b} and friends all vanish, leaving us with

$$S_2 = \sum_{(a,b)} 6 = 6N^2.$$

In total, we see that $S = \frac{1}{8} \cdot 4N^4 - \frac{1}{4} \cdot 6N^2 = \frac{N^4-3N^2}{2}$. Combining information from the other problems, we see $N = 5$, so we get $S = \boxed{275}$.

20. Let N be the answer to Problem 19, and let M be the last digit of N. Let ω be a primitive Mth root of unity, and define $P(x)$ such that

$$P(x) = \prod_{k=1}^{M} \left(x - \omega^{i_k}\right),$$

where the i_k are chosen independently and uniformly at random from the range $\{0, 1, \ldots, M-1\}$. Compute $\mathbb{E}\left[P\left(\sqrt{\lfloor \frac{1250}{N} \rfloor}\right)\right]$.

Answer: 32

Solution: Because the i_k are chosen independently and uniformly at random, for any fixed choice of x, the random variables $x - \omega^{i_k}$ are independent. Thus, we have that

$$\begin{aligned}\mathbb{E}\left[P(x)\right] &= \mathbb{E}\left[\prod_{k=1}^{M}(x-\omega^{i_k})\right] \\ &= \prod_{k=1}^{M}\mathbb{E}\left[(x-\omega^{i_k})\right] \\ &= \prod_{k=1}^{M}\left(x-\mathbb{E}\left[\omega^{i_k}\right]\right).\end{aligned}$$

However, for any k, note that

$$\mathbb{E}\left[\omega^{i_k}\right] = \frac{1}{M}\sum_{i=0}^{M-1}\omega^i = 0$$

because ω is a Mth root of unity.

Thus, $\mathbb{E}\left[P(x)\right] = x^M$, so

$$\mathbb{E}\left[P\left(\sqrt{\left\lfloor\frac{1250}{N}\right\rfloor}\right)\right] = \left(\sqrt{\left\lfloor\frac{1250}{N}\right\rfloor}\right)^M.$$

After combining all of the answers, we conclude that N is 275 so the answer to this problem is $\boxed{32}$.

21. Let N be the answer to Problem 20. Define the polynomial $f(x) = x^{34} + x^{33} + x^{32} + \cdots + x + 1$. Compute the number of primes $p < N$ such that there exists an integer k with $f(k)$ divisible by p.

Answer: 5

Solution: Observe that $f(x) = \frac{x^{35}-1}{x-1}$. Call a prime p *good* if and only if there exists an integer k such that $f(k) \equiv 0 \pmod{p}$. We claim that p is good if and only if $p \in \{5, 7\}$ or $p \equiv 1 \pmod{5}$ or $p \equiv 1 \pmod{7}$. We show the implications separately.

- Suppose there is some k with $f(k) \equiv 0 \pmod{p}$. Then note that $k^{35} \equiv 1 \pmod{p}$, so $k \pmod{p}$ has order dividing 35. However, if k has order 1, then $k \equiv 1 \pmod{p}$, so $f(k) \equiv 35 \pmod{p}$, so $p \in \{5, 7\}$.

  Thus, if $p \notin \{5, 7\}$, then the order of k is greater than 1 and dividing 35, so the order is in $\{5, 7, 35\}$. It follows that $p - 1$ is divisible by an element in $\{5, 7, 35\}$, so $p \equiv 1 \pmod{5}$ or $p \equiv 1 \pmod{7}$.
- Suppose $p \in \{5, 7\}$ or $p \equiv 1 \pmod{5}$ or $p \equiv 1 \pmod{7}$. If $p = 5$ or $p = 7$, then note that $f(1) = 35$ is divisible by p.

  Otherwise, select the $q \in \{5, 7\}$ with $p \equiv 1 \pmod{q}$. In this case, we let g be a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$, so g has order $p - 1$, so

  $$k = g^{(p-1)/q}$$

  has order q. It follows that $k^q - 1$ is divisible by p even though $k - 1$ is not divisible by p. Further, $k^{35} - 1$ is divisible by p, so $f(k) = \frac{k^{35}-1}{k-1}$ is divisible by p, which is what we wanted.

Thus, we want to count the number of primes $p < N$ with $p \in \{5, 7\}$ or $p \equiv 1 \pmod 5$ or $p \equiv 1 \pmod 7$.

We now figure out how to combine all of the answers. Let N_{19}, N_{20}, and N_{21} be the answers to Problem 19, 20, and 21, respectively. Notice that if $N_{19} > 1250$, then $N_{20} = 0$, and thus $N_{21} = 0$, which is not valid. Thus, we want $N_{19} \le 1250$. Because $N_{19} = \frac{N_{21}^4 - 3N_{21}^2}{2}$, we conclude that $N_{21} \le 7$. Moreover, because N_{21} represents the number of sides in a regular polygon, we need $N_{21} \ge 3$.

Thus, we can check all possibilities, as follows.

N_{21}	N_{19}	$\sqrt{\left\lfloor \frac{1250}{N_{19}} \right\rfloor}$	N_{20}	Matches?
3	27	$\sqrt{46}$	$46^{7/2}$	No
4	104	$\sqrt{12}$	144	No
5	275	2	32	Yes
6	594	$\sqrt{2}$	4	No
7	1127	1	1	No

We then conclude that the answer to this problem is $\boxed{5}$.

22. Set $n = 425425$. Let S be the set of proper divisors of n. Compute the remainder when

$$\sum_{k \in S} \varphi(k) \binom{2n/k}{n/k}$$

is divided by $2n$, where $\varphi(x)$ is the number of positive integers at most x that are relatively prime to it.

Answer: 390050

Solution: The key is to notice that this expression looks vaguely like it could be part of a Burnside's lemma application. We consider the number of ways to color the vertices of a regular $2n$-gon so that n vertices are red and n vertices are blue, where rotations are considered identical. Well, for every divisor k of n, there are $\varphi(k)$ rotations by $\frac{a}{k}$ of the circle where $\gcd(a, k) = 1$, and each such rotation fixes $\binom{2n/k}{n/k}$ colorings. Thus, by Burnside's lemma, there are

$$\frac{1}{2n} \sum_{k \mid n} \varphi(k) \binom{2n/k}{n/k}$$

such colorings. Because this is an integer, the sum $\sum_{k \mid n} \varphi(k) \binom{2n/k}{n/k}$ is divisible by $2n$. It follows

$$\begin{aligned} \sum_{k \in S} \varphi(k) \binom{2n/k}{n/k} &\equiv -\varphi(n) \binom{2}{1} \\ &\equiv -2\varphi(n) \\ &\equiv 2n - 2\varphi(n) \pmod{2n}. \end{aligned}$$

Plugging in $n = 425425 = 1001 \cdot 425 = 7 \cdot 11 \cdot 13 \cdot 5^2 \cdot 17$ gives $2n - 2\varphi(n) = \boxed{390050}$.

23. Carson the farmer has a plot of land full of crops in the shape of a 6×6 grid of squares. Each day, he uniformly at random chooses a row or a column of the plot that he hasn't chosen before and harvests all of the remaining crops in the row or column. Compute the expected number of connected components that the remaining crops form after 6 days. If all crops have been harvested, we say there are 0 connected components.

Answer: $\mathbf{\frac{115}{33}}$

Solution: For convenience, suppose we are working with an $n \times n$ grid and m days. Label the cells of the grid $(1,1)$ through (n,n), such that $(1,1)$ is the bottom-left corner.

Each of the connected components will be a rectangle; in particular, each one has a unique bottom-left corner. Thus, it is equivalent to count the expected number of cells of the grid that will become a bottom-left corner of some connected region by the end. Let $1_{i,j}$ be the indicator variable for cell (i,j) being a bottom-left corner of a region; i.e. $1_{i,j} = 1$ if (i,j) is the bottom-left corner of a region after 6 days, and $1_{i,j} = 0$ if otherwise. Thus, we want to compute

$$\mathbb{E}\left[\sum_{1 \leq i,j \leq n} 1_{i,j}\right] = \sum_{1 \leq i,j \leq n} \mathbb{E}[1_{i,j}],$$

by linearity of expectation. By the definition of $1_{i,j}$, each $\mathbb{E}[1_{i,j}]$ is equal to the probability that (i,j) is a bottom-left corner. There are $\binom{2n}{m}$ ways to select which rows and columns will disappear over the m days, so it remains to compute the number of ways they can disappear such that (i,j) becomes the bottom-left corner of some region, for each (i,j). There are three cases of similar logic to consider.

- The bottom-left corner $(1,1)$ of the entire grid becomes the bottom-left corner of some connected component if and only if both the column and row containing it do not disappear. Hence, all m must be chosen from the other $2n-2$ rows and columns, so there are $\binom{2n-2}{m}$ ways in this case.
- The second case is cells of the form $(1,j)$ or $(i,1)$, where $2 \leq i,j \leq n$. There are $2(n-1)$ cells of this form. Consider one of the $(1,j)$: this cell becomes the bottom-left corner of some region if and only if the row and column containing it do not disappear, and the row below it does disappear. Then, the other $m-1$ disappearing rows and columns must be chosen from the other $2n-3$ rows and columns, so there are $\binom{2n-3}{m-1}$ ways for one of the $(1,j)$ to become a bottom-left corner.

  The same holds for all cells of the form $(i,1)$: the situation is the same but just that the column to the left of the cell necessarily disappears, instead of the row below.
- The final case is for cells of the form (i,j) with $2 \leq i,j \leq n$. There are $(n-1)^2$ such cells, and each of these (i,j) becomes a bottom-left corner if and only if the row and column containing it do not disappear, and the row below it and the column to the left of it both disappear. Then, the remaining $m-2$ disappearing rows and columns must be chosen from the other $2n-4$, for which there are $\binom{2n-4}{m-2}$ ways.

Thus, summing over all of the cells gives the answer as

$$\frac{\binom{2n-2}{m} + 2(n-1)\binom{2n-3}{m-1} + (n-1)^2\binom{2n-4}{m-2}}{\binom{2n}{m}}.$$

Plugging in $n = 6$ and $m = 6$ and evaluating gives $\boxed{\frac{115}{33}}$.

24. Let $\triangle BCD$ be an equilateral triangle and A be a point on the circumcircle of $\triangle BCD$ such that A is on the minor arc $\widehat{BD}$. Then, let P be the intersection of $\overline{AB}$ with $\overline{CD}$, Q be the intersection of $\overline{AC}$ with $\overline{DB}$, and R be the intersection of $\overline{AD}$ with $\overline{BC}$. Finally, let X, Y, and Z be the feet of the altitudes from P, Q, and R, respectively, in triangle $\triangle PQR$. Given $BQ = 3-\sqrt{5}$ and $BC = 2$, compute the product of the areas $[\triangle XCD]\cdot[\triangle YDB]\cdot[\triangle ZBC]$.

Answer: $\frac{3\sqrt{3}}{16}$

Solution: Let O be the circumcenter of $(ABCD)$. By the Miquel point theorem for cyclic quadrilaterals, O is the orthocenter of $\triangle PQR$ and the feet of the altitudes X, Y, and Z are the three Miquel points with respect to $ABCD$. Therefore, X, Y, and Z are concyclic with the following circles:

$$\begin{aligned}
X&\colon\ (QAD),\ (QBC),\ (RAC),\ (RBD),\\
Y&\colon\ (RAB),\ (RCD),\ (PAD),\ (PCB),\\
Z&\colon\ (PAC),\ (PDB),\ (QAB),\ (QDC).
\end{aligned}$$

We claim that the following sets of triangles are similar:

$$\begin{aligned}
&\triangle BQR \sim \triangle XDC \sim \triangle XCR \sim \triangle XQD,\\
&\triangle CRP \sim \triangle YBD \sim \triangle YDP \sim \triangle YRB,\\
&\triangle DPQ \sim \triangle ZCB \sim \triangle ZBQ \sim \triangle ZPC,
\end{aligned}$$

and will show this by directed angle chasing, in order to preserve independency of configuration. We have

$$\begin{aligned}
\measuredangle QRB &= \measuredangle XRA + \measuredangle ARC\\
&= \measuredangle XCA + \measuredangle ACR + \measuredangle RAC\\
&= \measuredangle XCR + \measuredangle DAC\\
&= \measuredangle XCB + \measuredangle BCD\\
&= \measuredangle XCD,
\end{aligned}$$

so by symmetry we also have $\measuredangle RQB = \measuredangle XDC$, meaning $\triangle BQR \sim \triangle XDC$. Because $BCXQ$ and $BDXR$ are cyclic, we also have $\triangle BQR \sim \triangle XCR \sim \triangle XQD$, ultimately implying that $\triangle BQR \sim \triangle XDC \sim \triangle XCR \sim \triangle XQD$ as desired. By a similar argument, we can produce the other two sets of similarities.

Letting s denote the side length of equilateral triangle $\triangle BCD$, we can find the product of areas using Law of Sines to be:

$$\begin{aligned}
[\triangle XCD]\cdot[\triangle YDB]\cdot[\triangle ZBC] &= \prod_{\text{cyc}}[\triangle XCD]\\
&= \prod_{\text{cyc}}\left([\triangle BQR]\cdot\left(\frac{CD}{QR}\right)^2\right)\\
&= \left(\frac{s^2\sqrt{3}}{4}\right)^3\prod_{\text{cyc}}\frac{BQ\cdot RB}{QR^2}.
\end{aligned}$$

Let B' be the reflection of B over $\overline{CD}$, and define C' and D' similarly. We claim that Q, R, and B' are collinear, with a symmetric property following with respect to C and D. Let

B^* be the intersection of $\overline{QR}$ with the circumcircle of $\triangle CDB'$. Angle chasing, we see that $\measuredangle DXR = \measuredangle DXQ = \measuredangle RXC$, meaning $\overline{RX}$ bisects angle $\angle CXD$. Therefore, $\overline{RX}$ must pass through the midpoint of major arc $\widehat{CD}$ of $\odot(CDB')$, namely B'. Therefore, $B^* = B'$, meaning R, X, B' are collinear and thus Q, R, B' are collinear.

Now, because $\overline{BQ}$ and $\overline{CB'}$ are parallel, it follows that $\triangle BQR \sim \triangle CB'R$ and by similar reasoning, $\triangle BQR \sim \triangle DQB'$. Thus, we can simplify to

$$\begin{aligned}\left(\frac{s^2\sqrt{3}}{4}\right)^3 \prod_{\text{cyc}} \frac{BQ \cdot RB}{QR^2} &= \left(\frac{s^2\sqrt{3}}{4}\right)^3 \prod_{\text{cyc}} \frac{BQ}{QR} \cdot \frac{RB}{QR} \\ &= \left(\frac{s^2\sqrt{3}}{4}\right)^3 \prod_{\text{cyc}} \frac{CB'}{B'R} \cdot \frac{B'D}{QB'} \\ &= \left(\frac{s^4\sqrt{3}}{4}\right)^3 \prod_{\text{cyc}} \frac{1}{QB' \cdot B'R}.\end{aligned}$$

Next, because $\frac{PC'}{C'R} = \frac{PD}{DC} = \frac{PQ}{QD'}$, we see $\overline{C'Q}$ and $\overline{D'R}$ are parallel. Applying symmetry, we obtain $\overline{B'P} \parallel \overline{C'Q} \parallel \overline{D'R}$ and therefore, the three corresponding lines are concurrent at a point at infinity. Furthermore, B' lies on QR, C' lies on RP, and D' lies on PQ meaning we can apply Ceva's Theorem. This gets us

$$\frac{QB'}{B'R} \cdot \frac{RC'}{C'P} \cdot \frac{PD'}{D'Q} = 1,$$

so

$$\prod_{\text{cyc}} QB' = \prod_{\text{cyc}} B'R,$$

allowing us to further simplify to

$$[\triangle XCD] \cdot [\triangle YDB] \cdot [\triangle ZBC] = \left(\frac{s^4\sqrt{3}}{4}\right)^3 \left(\prod_{\text{cyc}} QB'\right)^{-2} = \left(\frac{s^2\sqrt{3}}{4}\right)^3 \left(\prod_{\text{cyc}} \frac{QB'}{s}\right)^{-2}.$$

Then, denote $\frac{QD}{s}$ with k, meaning $k = \frac{\sqrt{5}-1}{2}$. Then, we can chase for the desired ratios: note $\frac{QB'}{s} = \frac{QB'}{B'D} = \sqrt{k^2+k+1}$, and $\frac{RC'}{s} = \frac{RC'}{C'B} = \sqrt{\left(\frac{1-k}{k}\right)^2 - \left(\frac{1-k}{k}\right) + 1}$ because $\frac{RB}{s} = \frac{RB}{B'D} = \frac{1-k}{k}$, and $\frac{PD'}{s} = \frac{PD'}{D'C} = \sqrt{\left(\frac{1}{1-k}\right)^2 + \left(\frac{1}{1-k}\right) + 1}$ because $\frac{PC}{s} = \frac{PC}{BD'} = \frac{DB}{QB} = \frac{1}{1-k}$. Noting that

$1-k=k^2$ and $k+\frac{1}{k}=\sqrt{5}$ we get

$$\begin{aligned}
[\triangle XCD]\cdot[\triangle YDB]\cdot[\triangle ZBC] &= \left(\frac{s^2\sqrt{3}}{4}\right)^3\left(\prod_{\text{cyc}}\frac{QB'}{s}\right)^{-2} \\
&= \frac{\left(\frac{s^2\sqrt{3}}{4}\right)^3}{(k^2+k+1)\left[\left(\frac{1-k}{k}\right)^2-\left(\frac{1-k}{k}\right)+1\right]\left[\left(\frac{1}{1-k}\right)^2+\left(\frac{1}{1-k}\right)+1\right]} \\
&= \frac{\left(\frac{s^2\sqrt{3}}{4}\right)^3}{\left(k+1+\frac{1}{k}\right)\left(k-1+\frac{1}{k}\right)\left(\frac{1}{k^2}+1+k^2\right)} \\
&= \frac{3\sqrt{3}}{(\sqrt{5}+1)(\sqrt{5}-1)4} \\
&= \boxed{\frac{3\sqrt{3}}{16}}.
\end{aligned}$$

25. For triangle $\triangle ABC$, define its A-*excircle* to be the circle that is externally tangent to line segment $\overline{BC}$ and extensions of $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, and define the B-excircle and C-excircle likewise. Then, define the A-*veryexcircle* to be the unique circle externally tangent to both the A-excircle as well as the extensions of $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, but that shares no points with line $\overleftrightarrow{BC}$, and define the B-veryexcircle and C-veryexcircle likewise.

Compute the smallest integer $N\geq 337$ such that for all $N_1\geq N$, the area of a triangle with lengths $3N_1^2$, $3N_1^2+1$, and $2022N_1$ is at most $\frac{1}{22022}$ times the area of the triangle formed by connecting the centers of its three veryexcircles. If your submitted estimate is a positive number E and the true value is A, then your score is given by $\max\left(0,\left\lfloor 25\min\left(\frac{E}{A},\frac{A}{E}\right)^3\right\rfloor\right)$.

Answer: 218966

Solution: The main idea here is that, for very large N_1^2, the value $3N_1^2+1\approx 3N_1^2$ grows much more quickly than $2022N_1 > 6\sqrt{N_1^2}$, so although the triangle is acute, two of the angles of the triangle are approximately $90°$. Let A, B, and C be the vertices across from the sides of length $2022N_1$, $3N^2$, and $3N^2+1$, respectively, and let A_v, B_v, and C_v be the three veryexcenters. The area of $\triangle ABC$ is around $\Delta=\frac{1}{2}\left(3N_1^2\right)(2022N_1)$ and the incircle, A-excircle, and A-veryexcircle have diameters of about $2022N_1$ (which we can even approximate to 0 for large N_1).

Next, the B-exradius can be calculated as

$$\frac{\Delta}{s-3N_1^2}\approx\frac{(3N_1^2)(2022N_1)/2}{2022N_1/2}=3N_1^2,$$

and the same logic holds for the C-exradius. Furthermore, using the estimation $\angle ABC, \angle ACB\approx 90°$, the ratio between the B-veryexradius and B-exradius is around $\frac{\sqrt{2}+1}{\sqrt{2}-1}=(\sqrt{2}+1)^2$, implying that the B and C-veryexradii are approximately $3(\sqrt{2}+1)^2N_1^2$.

The line from the incenter to the B-veryexcenter will approximately be the line bisecting the C-exterior angle, and same with the C-veryexcenter and B-exterior angle, and we can also approximate A_v to be the incenter, giving us that $\angle B_vA_vC_v\approx\frac{90°+90°}{2}=90°$. So, this triangle will be approximately a 45-45-90 triangle. Lastly, the distance between B_v and C_v is approximately the sum of the two veryexradii, or $6(\sqrt{2}+1)^2N_1^2$. Thus, the area of $\triangle A_vB_vC_v$ is approximately equal to $\frac{1}{4}\left(6(\sqrt{2}+1)^2N_1^2\right)^2 = 9(\sqrt{2}+1)^4N_1^4$.

For our approximation of N, we would like for $9(\sqrt{2}+1)^4N^4/\Delta = 22022$. Plugging in the approximation of Δ, we get

$$N \approx \frac{1011 \cdot 22022}{3(\sqrt{2}+1)^4} \approx 218466,$$

which comfortably nets a total score of 24 points. Making small approximations for values like $3(1+\sqrt{2})^4 \approx 100, (1+\sqrt{2})^2 \approx 6$, or $1011 \approx 1000$ still keeps the score at around 20 points.

26. Compute the number of positive integers n less than 10^8 such that at least two of the last five digits of

$$\left\lfloor 1000\sqrt{25n^2 + \frac{50}{9}n + 2022} \right\rfloor$$

are 6. If your submitted estimate is a positive number E and the true value is A, then your score is given by $\max\left(0, \left\lfloor 25\min\left(\frac{E}{A}, \frac{A}{E}\right)^7 \right\rfloor\right)$.

Answer: 37040

Solution: We can rewrite the expression as

$$\left\lfloor 1000\sqrt{\left(5n+\frac{5}{9}\right)^2 - \frac{25}{81} + 2022} \right\rfloor.$$

Note that as n becomes very large, $\left\lfloor 1000\sqrt{\left(5n+\frac{5}{9}\right)^2 - \frac{25}{81} + 2022} \right\rfloor$ converges to $\left\lfloor 1000(5n+\frac{5}{9}) \right\rfloor$, which ends in an infinite number of trailing 5s. We care about the last five digits of this number, which are the tens, ones, tenths, hundredths, and thousandths digits of $\sqrt{25n^2 + \frac{50}{9}n + 2022}$.

Now, note that as n grows large, the tens digit will always cycle through each of 0 to 9, so 6 will appear around a tenth of the time. Moreover, the integer part of the number will become a multiple of 5, so the ones digit will alternate between 0 and 5. However, all of the other digits will tend towards 5 from above. So, we bound when $\sqrt{\left(5n+\frac{5}{9}\right)^2 - \frac{25}{81} + 2022}$ is slightly larger than $5n + \frac{5}{9}$.

- $\sqrt{25n^2 + \frac{50}{9}n + 2022} - 5n \in [1, 2)$:

  Solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 1$, we have:

$$25n^2 + \frac{50}{9}n + 2022 = 25n^2 + 10n + 1$$
$$454 \geq n.$$

  Then, solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 2$, we have:

$$25n^2 + \frac{50}{9}n + 2022 = 25n^2 + 20n + 4$$
$$140 \leq n.$$

  So, this case gives us around $\frac{454-140+1}{20}$ values.

- $\sqrt{25n^2 + \frac{50}{9}n + 2022} - 5n \in [0.6, 0.7)$:

  Solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 0.6$, we get that $4548 \geq n$.

  Then, solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 0.7$, we get that $1400 \leq n$. So, this case gives us around $\frac{4548-1400+1}{10}$ values.

- $\sqrt{\left(5n + \frac{5}{9}\right)^2 - \frac{25}{81} + 2022} - 5n \in [0.56, 0.57)$:
  From the previous cases, we can guess that the bounds on n are going to be around 14000 and 45460. However, we solve for n for sake of completeness.
  Solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 0.56$, we get that $45487 \geq n$.

  Then, solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 0.57$, we get that $13997 \leq n$. This case gives us around $\frac{45487-13997+1}{10}$ values.

- $\sqrt{\left(5n + \frac{5}{9}\right)^2 - \frac{25}{81} + 2022} - 5n \in [0.556, 0.557)$:
  Again, we can guess that the bounds on n are going to be around 139970 and 454870. However, we solve for n for sake of completeness.
  Solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 0.556$, we get that $454880 \geq n$.

  Then, solving for $\sqrt{25n^2 + \frac{50}{9}n + 2022} = 5n + 0.557$, we get that $139964 \leq n$. This case gives us around $\frac{454880-139964+1}{10}$ values.

Considering these cases alone gives $\frac{454-140+1}{20} + \frac{4548-1400+1}{10} + \frac{45487-13997+1}{10} + \frac{454880-139964+1}{10} \approx$ 34971 solutions, which would give us 16 points. A less computational way to get around the same result is to recognize, given the properties of square roots, that each case gives 10 times the number of solutions as the last, for an estimate of $\frac{454-140+1}{20} \cdot (1 + 20 + 200 + 2000) \approx 34981$ solutions, which also gives 16 points. Indeed, just noticing that the final case will dominate and getting an estimate of $\frac{454880-139964+1}{10} \approx 31492$ solutions gives 8 points.

To further improve our score, we may consider $\sqrt{25n^2 + \frac{50}{9}n + 2022} - 5n \in [0.566, 0.567)$ because there will always be two 6s, no matter the digits dictated by the $5n$. We could guess that there are a little less than $\frac{45487-13997+1}{10}$ such cases, for a total of $\frac{9}{10} \cdot \frac{45487-13997+1}{10} \approx 2834$ more solutions, leading to an estimate of 37805 for a score of 21 points. Solving the system or getting a better estimate for this case would potentially grant even more points.

27. Submit a positive integer n less than 10^5. Let the sum of the valid submissions from all teams to this question be S. If you submit an invalid answer, you will receive 0 points. Otherwise, your score will be $\max\left(0, \left\lfloor 25 - \frac{|S'-n|}{10} \right\rfloor\right)$, where S' is the sum of the squares of the digits of S.

    Answer: N/A

    Solution: Congratulations, you got through the Guts round! We hope you enjoyed the test, as well as the rest of BMT!

    There is some strategy for approaching this problem. Note that the maximum possible value of S is less than (# of teams) $\times 10^5$. There are around 100–200 teams, so $S < 10^9$, and the maximum value of S' can take on is $9 \cdot 81 = 729$, but in reality, in this case S would be around $9 \cdot \frac{0^2+1^2+2^2+\cdots+9^2}{10} = 256.5$ or even less if all teams submit and guess completely randomly. Moreover, not all teams are going to submit, and some teams will recognize that there is upper bound on S' and will therefore submit a 3-digit answer. This would give a rough ballpark of approximately 6 or 7 random digits, so a good guess would be around $6 \cdot \frac{0^2+1^2+2^2+\cdots+9^2}{10} = 171$ or $7 \cdot \frac{0^2+1^2+2^2+\cdots+9^2}{10} = 199.5$, give or take.

Here are some statistics from our in-person contest and satellites:

- BMT: $S = 321604, S' = 66$.
- BMT Toronto: $S = 60853, S' = 134$.
- BMT China: $S = 118064, S' = 118$.